Field-induced quantum critical point in planar Heisenberg ferromagnets with long-range interactions: Two-time Green's function framework

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The two-time Green's function method is used to study the critical properties and crossover phenomena near the field-induced quantum critical point (QCP) of a d-dimensional spin-S planar Heisenberg ferromagnet with long-range interactions decaying as $r^{-\alpha}$ (with $\alpha > d$) with the distance r between spins. We adopt the Callen scheme for spin S and the Tyablikov decoupling procedure which is expected to provide suitable results at low temperatures. Different quantum critical regimes are found in the (α, d) plane and the global structure of the phase diagram is determined showing the typical V-shaped region close to the QCP. Depending on the values of α, we find that also for dimensionalities $d \leqslant 2$ a finite-temperature critical line, ending in the QCP, exists with asymptotic behaviors and crossovers which can be employed as a useful guide for experimental studies. Moreover, these crossovers are shown to be suitably described in terms of (α, d)-dependent scaling functions and effective critical exponents.

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I. INTRODUCTION

The study of the macroscopic properties of low-dimensional quantum magnetic systems is one of the most active research subjects in condensed matter physics.[1–14] In particular, field-induced quantum phase transitions (QPTs) and anisotropy effects have been the focus of a great amount of theoretical and experimental investigations.[4–7,15–22] Today, quantum critical points (QCPs) (Refs. 15–17) are currently considered responsible for the unusual low-temperature properties observed in many low-dimensional magnetic compounds. Moreover, there are also clear evidences that magnetic anisotropies of different nature play a fundamental role to lead new interesting phenomena.[23,24]

In this context, anisotropic Heisenberg models have attracted increasing attention as a relevant tool to understand many relevant properties of a lot of magnetic materials. In particular, from the perspective of exchange interaction anisotropy, the *XXX*, *XXZ*, and *XYZ* models provide a good starting platform for exploring the magnetic response of several types of low-dimensional ferromagnets (FMs) and antiferromagnets.

Also the range of interactions are found to play a crucial role in the low-dimensional magnetism. For the isotropic Heisenberg model with short-range interactions (SRIs), the Mermin-Wagner theorem[25] excludes long-range order at finite temperature for dimensionalities $d \leqslant 2$. However, such a type of idealized systems does not exist in nature and many observed properties cannot be explained by means of this simple model.

In real metallic magnetic compounds, as most ultrathin films investigated experimentally consisting of metals or alloys, the exchange interactions are found to be of the Ruderman-Kittel-Kasuya-Yoshida type,[26] which have a long-range oscillatory behavior for large distance between spins.

Rigorous statements about one- and two-dimensional magnetic systems of this type were performed by Bruno[27] by means of an extension of the Mermin-Wagner theorem to anisotropic Heisenberg magnets to include long-range interactions (LRIs) decreasing as $r^{-\alpha}$ with the distance r between spins, with a possible oscillatory character.

As a general rule, LRIs are always of interest since they can induce unusual behaviors. In particular they can drastically influence the critical behavior in low-dimensional magnetic systems. Hence, microscopic investigations of the LRIs effects are particularly relevant in the quantum theory of magnetism.

Heisenberg models with LRIs have attracted great attention since the exact ground state of the spin-1/2 Heisenberg with interactions decaying as r^{-2} was found independently by Haldane[28] and Shastry.[29]

For isotropic Heisenberg FMs with LRIs of the $r^{-\alpha}$ type, the critical scenario in d dimensions (with $\alpha > d$) is qualitatively clear both for quantum[30] and classical[31–33] systems.

In contrast, few results have been obtained microscopically for anisotropic spin-S Heisenberg models with LRIs which may also exhibit field-induced QPTs,[3,4,9] especially regarding the critical properties and crossovers close to a QCP.

As for the usual isotropic and anisotropic Heisenberg models with SRIs, one expects that a variety of interesting results on microscopic grounds can be obtained also for this type of magnetic systems by employing the powerful two-time Green's function (GF) method,[34–36] which may provide results better than the mean-field (MF) approximation. Essentially, one obtains an equation of motion (EM) for the two-time GF of interest where higher order GFs occur. For each of them one again writes down an additional EM and so on, producing an infinite hierarchy of coupled equations. To find tractable solutions, decoupling procedures are usually necessary to truncate this infinite equations chain at a given level to obtain a closed set of self-consistent equations for thermodynamic averages. For magnetic systems, one currently adopts the Tyablikov decoupling (TD),[35] also known as the random phase approximation (RPA), which neglects the correlations between longitudinal and transverse spin fluctuations. This is considered the simplest and most reliable decoupling scheme and is still used for suitable analytical studies in the theory of magnetism.[2] However, its accuracy has been often brought into question (see, for instance, Ref. 21). Several higher-order decoupling procedures have been developed but no concrete improvements have been achieved about criticality. In this sense the TD is still considered the first step of approximation providing qualitative, and sometimes quantitative, correct physics for a variety of experimentally interesting situations in the low-temperature limit.[37] In particular, it seems especially effective in exploring quantum criticality providing also nearly exact results sufficiently close to a QCP.[38]

In this paper we use the two-time GF method adopting the Callen approach[39] for spin S, within the TD scheme to investigate the low-temperature static and dynamic properties of a d-dimensional spin-S easy-plane Heisenberg FM in the presence of a longitudinal magnetic field[38,40,41] with anisotropic LRIs decaying as $r^{-\alpha}$. As we will see, for $\alpha > d$ and any d, this model exhibits a field-induced QCP around which the longitudinal magnetization is very near the saturation value S characterizing the full polarized state. This feature makes the TD particularly effective in investigating the physics of the model in the influence region of the QCP.

Our plan consists in (i) exploring the effects of (low-temperature) anisotropy and applied magnetic field for arbitrary spin S and dimensionality d when long-range interactions are involved and (ii) deriving, microscopically, the quantum critical scenario which takes place at low-temperature close to the QCP.

The paper is organized as follows. In Sec. II, we first introduce the model of interest and discuss its physical relevance. We next present the basic equations within the two-time GF formalism combining the Callen method[39] for spin S and the Tyablikov decoupling[35] in the equation of motion for the appropriate GF. Here, the basic self-consistent equations for exploring the quantum criticality are also presented and the relevant thermodynamic quantities are defined. Section III is devoted to determine the field-induced QCP and the finite temperature critical line (CL) (when it exists) for transverse ordering by variation of the dimensionality d and the interaction parameter α (with $\alpha > d$). Numerical results for the phase boundary are also presented for different values of S and special choices of d and α in the region of the (α, d) plane where the LRIs are expected to be effective. The static quantum critical properties and crossovers near the QCP are analyzed in Sec. IV in a systematic way. Section V is dedicated to a scaling function description of crossovers within a relevant region of the (α, d) plane. Finally, in Sec. VI, remarks on quantum critical dynamics are performed and some conclusions are drawn. Useful mathematical ingredients are collected in Appendixes A–C.

II. SPIN MODEL AND BASIC EQUATIONS WITHIN THE TWO-TIME GREEN'S FUNCTION FRAMEWORK

We consider the general anisotropic Heisenberg ferromagnetic model with Hamiltonian (in convenient units)

$$\mathcal{H} = -\frac{1}{2}\sum_{i,j=1}^{N} [J_{ij}(S_i^x S_j^x + S_i^y S_j^y) + K_{ij} S_i^z S_j^z] - h\sum_{i=1}^{N} S_i^z, \quad (2.1)$$

where $\mathbf{S}_i \equiv (S_i^x, S_i^y, S_i^z)$ is the spin-S operator at site i of a d-dimensional periodic hypercubic lattice with unity spacing, N is the number of sites, $J_{ij} = J(|\mathbf{r}_i - \mathbf{r}_j|) \geq 0$ and $K_{ij} = K(|\mathbf{r}_i - \mathbf{r}_j|) \geq 0$ (with $J_{ii} = K_{ii} = 0$) are the in-plane and longitudinal exchange interactions between sites i and j, and $h \geq 0$ is a magnetic field applied along the z direction. Here $\mathbf{S}_i^2 = S(S+1)$ and the spin component $S_i^\alpha (\alpha = x, y, z)$ satisfy the commutation relations $[S_i^\alpha, S_j^\beta] = i\varepsilon_{\alpha\beta\gamma} S_i^\gamma \delta_{ij}$, $\varepsilon_{\alpha\beta\gamma}$ being the usual Levi-Civita symbol.

We focus here on the so-called easy-plane or planar Heisenberg ferromagnet (PFM) with $K_{ij} < J_{ij}$,[20,40,42–47] which has attracted recently a renewed interest[4–7,20,21,48] due to the possible existence of complex magnetic compounds, exhibiting field-induced QPTs,[49–52] whose intricate microscopic Hamiltonian can be mapped[46] into an effective one of the simple form (2.1). RG studies of quantum criticality for the spin-1/2 PFM with short-range interactions have been recently performed.[18–20] The same spin model was explored, although not in the context of the modern view of quantum critical scenario, by Brown[41] via a GFs approach involving an approximation scheme which is based on the assumption of the statistical independence of the ordering operators. Quite recently,[38,53] the quantum critical properties of a simplified version of model (2.1), with in-plane short-range interactions and uniform longitudinal coupling $K_{ij} = K/N$, was explored for spin 1/2 in the influence domain of the field-induced QCP by using the two-time GFs equations of motion method within the TD scheme which is expected to provide nearly exact results in the low-temperature regime close to the full polarized state.[38] However, a complete microscopic description of the quantum criticality of the PFM for arbitrary dimensionality d and spin S with long-range power-law interactions has not been performed up to now, especially regarding low-dimensional ferromagnetic systems. This will be the objective of the following developments via the two-time GFs method.

Following the Callen procedure,[39] we now introduce the retarded two-time commutator GF,

$$G_{ij}(t-t') = \langle\langle S_i^+(t); e^{aS_j^z} S_j^-(t')\rangle\rangle = -i\theta(t-t')\langle[S_i^+(t-t'), e^{aS_j^z} S_j^-]\rangle. \quad (2.2)$$

Here, $A(t) = e^{i\mathcal{H}t} A e^{-i\mathcal{H}t}$, $\theta(t-t')$ is the step function, $S_j^\pm = S_j^x \pm iS_j^y$ are the raising $(+)$ and lowering $(-)$ spin operators, and $\langle \cdots \rangle = \mathrm{Tr}\{(\cdots)e^{-\beta\mathcal{H}}\}/\mathrm{Tr}\, e^{-\beta\mathcal{H}}$, where β denotes the inverse of temperature T. In Eq. (2.2) a is the Callen auxiliary parameter to be properly set equal to zero at the end of calculations.

The equation of motion for the GF (2.2) in the ω-Fourier representation[35] becomes

$$(\omega - h)\langle\langle S_i^+ | e^{aS_j^z} S_j^-\rangle\rangle_\omega = \Psi(a)\delta_{ij} - \sum_h J_{hi}\langle\langle S_i^z S_h^+ | e^{aS_j^z} S_j^-\rangle\rangle_\omega + \sum_h K_{hi}\langle\langle S_h^z S_i^+ | e^{aS_j^z} S_j^-\rangle\rangle_\omega, \quad (2.3)$$

where $\langle\langle A|B\rangle\rangle_\omega = \int_{-\infty}^{+\infty} dt e^{i\omega t}\langle\langle A(t); B\rangle\rangle$ and

$$\Psi(a) = \langle [S_i^+, e^{aS_i^z} S_i^-] \rangle, \quad (2.4)$$

which does not depend on the site index due to the space translational invariance.

Notice that one has $\Psi(0)=\langle [S_i^+,S_i^-]\rangle = 2m$, where $m=\langle S_i^z\rangle = \sigma S$, with $0 \leqslant \sigma \leqslant 1$ is the longitudinal magnetization per spin. We will name $\sigma = m/S$ of the reduced magnetization. Now, to determine the GF of interest, we perform in Eq. (2.3) the TD,[35]

$$\langle\langle S_h^z S_k^+ | e^{aS_j^z} S_j^- \rangle\rangle_\omega \simeq \langle S_h^z \rangle \langle\langle S_k^+ | e^{aS_j^z} S_j^- \rangle\rangle_\omega. \quad (2.5)$$

Thus, the equation of motion (2.3) reduces to

$$(\omega - h - mK(0)) G_{ij}(\omega) = \Psi(a)\delta_{ij} - m \sum_h J_{hi} G_{hj}(\omega), \quad (2.6)$$

where $K(0)=\Sigma_h K_{hi} = \Sigma_{\mathbf{r}} K(|\mathbf{r}|) e^{i\mathbf{k}\cdot\mathbf{r}}|_{\mathbf{k}=0}$ is the zero-wave-vector Fourier component of K_{ij} and $G_{ij}(\omega) \equiv \langle\langle S_i^+ | e^{aS_j^z} S_j^- \rangle\rangle_\omega$.

Finally, Eq. (2.6) can be easily solved in the $\mathbf{k}$-wave-vector space [with $G_{ij}(\omega) = \frac{1}{N}\Sigma_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{r}_i-\mathbf{r}_j)} G_{\mathbf{k}}(\omega)$] and the solution reads (with $\omega \to \omega + i\varepsilon$, $\varepsilon \to 0^+$, for the retarded GF)

$$G_{\mathbf{k}}(\omega) = \frac{\Psi(a)}{\omega - \omega_{\mathbf{k}}}, \quad (2.7)$$

where

$$\omega_{\mathbf{k}} \equiv \omega_{\mathbf{k}}(\sigma) = \omega_0(\sigma) + S\sigma[J(0) - J(\mathbf{k})] \quad (2.8)$$

represents the energy spectrum of undamped excitations in the system,

$$\omega_0(\sigma) = h + S\sigma[K(0) - J(0)] \quad (2.9)$$

is the related energy gap and $J(\mathbf{k}) = \Sigma_{\mathbf{r}} J(|\mathbf{r}|) e^{i\mathbf{k}\cdot\mathbf{r}}$. Here the wave-vectors $\mathbf{k}$ range in the first Brillouin zone (1BZ).

The next crucial step is to determine the function $\Psi(a)$ and hence $\sigma = \Psi(0)/2S$. For utility of the reader, we outline below the Callen procedure for solving this problem.

In terms of the spectral density $\Lambda_{ij}(\omega) = \frac{1}{N}\Sigma_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{r}_i-\mathbf{r}_j)} \Lambda_{\mathbf{k}}(\omega) = 2\pi\Psi(a)\frac{1}{N}\Sigma_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{r}_i-\mathbf{r}_j)} \delta(\omega-\omega_{\mathbf{k}})$ of the GF $G_{ij}(\omega)$, the spectral theorem[35] provides the representation

$$\langle e^{aS_j^z} S_j^- S_i^+ \rangle = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{\Lambda_{ij}(\omega)}{e^{\beta\omega} - 1} = \Psi(a) \frac{1}{N} \sum_{\mathbf{k}} \frac{e^{i\mathbf{k}\cdot(\mathbf{r}_i-\mathbf{r}_j)}}{e^{\beta\omega_{\mathbf{k}}} - 1}. \quad (2.10)$$

In particular, we have

$$\langle e^{aS_i^z} S_i^- S_i^+ \rangle = \Psi(a)\Phi, \quad (2.11)$$

where

$$\Phi = \Phi(T,h;\sigma) = \frac{1}{N}\sum_{\mathbf{k}} \frac{1}{e^{\beta\omega_{\mathbf{k}}(\sigma)} - 1} \overset{N\to\infty}{=} \int_{1BZ} \frac{d^d k}{(2\pi)^d} \frac{1}{e^{\beta\omega_{\mathbf{k}}(\sigma)} - 1}. \quad (2.12)$$

Then, by using the identity $[S_i^+, (S_i^z)^n] = \{(S_i^z - 1)^n - (S_i^z)^n\} S_i^+$ and the expansion $e^{aS_i^z} = \Sigma_{n=0} (a^n/n!) (S_i^z)^n$, a simple algebra yields[39]

$$\Psi(a) = S(S+1)(e^{-a} - 1)\Omega(a) + (e^{-a} + 1)\Omega'(a) - (e^{-a} - 1)\Omega''(a), \quad (2.13)$$

with

$$\Omega(a) = \langle e^{aS_i^z} \rangle. \quad (2.14)$$

Since we have $m = \frac{1}{2}\Psi(0) = d\Omega(a)/da|_{a=0}$, it becomes now necessary to determine the function $\Omega(a)$. Using the exact operatorial relation $S_i^- S_i^+ = S(S+1) - S_i^z - (S_i^z)^2$ and Eq. (2.11) we get for $\Omega(a)$ the differential equation,[39,54]

$$\Omega''(a) + \frac{(1+\Phi)e^a + \Phi}{(1+\Phi)e^a - \Phi} \Omega'(a) - S(S+1)\Omega(a) = 0, \quad (2.15)$$

to be solved with the boundary conditions,

$$\Omega(0) = 1, \quad \prod_{p=-S}^{S} \left(\frac{d}{da} - p \right) \Omega(a)|_{a=0} = 0. \quad (2.16)$$

The solution is known[39,54] and given by

$$\Omega(a) = \frac{(1+\Phi)^{2S+1} e^{(S+1)a} - \Phi^{2S+1} e^{-Sa}}{[(1+\Phi)^{2S+1} - \Phi^{2S+1}][(1+\Phi)e^a - \Phi]}. \quad (2.17)$$

It is now easy to obtain an explicit expression of the reduced magnetization $\sigma = m/S$ in terms of Φ for arbitrary S. One finds

$$\sigma = 1 - \frac{\Phi}{S} + \frac{(2S+1)/S}{(1+\Phi^{-1})(2S+1) - 1} \equiv B_S(\Phi), \quad (2.18)$$

where Φ is expressed as a function of σ itself through Eq. (2.12).

Equations (2.8), (2.12), and (2.18) constitute a set of self-consistent equations to determine the reduced magnetization σ as a function of T and h, and hence $G_{\mathbf{k}}(\omega)$ given by Eq. (2.7).

A key quantity for our purposes is the transverse GF

$$G_\perp(\mathbf{k},\omega) = \langle\langle S_i^+ | S_j^- \rangle\rangle_{\omega,\mathbf{k}} \equiv G_{\mathbf{k}}(\omega)|_{a=0}. \quad (2.19)$$

Within the TD scheme, Eq. (2.7) provides

$$G_\perp(\mathbf{k},\omega) = \frac{2S\sigma}{\omega - \omega_{\mathbf{k}}(\sigma)}. \quad (2.20)$$

This allows us to determine the static transverse susceptibility as

$$\chi_\perp(T,h) = -G_\perp(0,0) = \frac{2S\sigma}{\omega_0(\sigma)} = \frac{2S\sigma(T,h)}{h + S\sigma(T,h)[K(0) - J(0)]}. \quad (2.21)$$

Of course, the longitudinal susceptibility will be given by $\chi_\parallel(T,h) = S(\partial\sigma(T,h)/\partial h)$.

It is worth noting that the stability condition $\chi_\perp \geqslant 0$ requires that $h + S\sigma(T,h)[K(0) - J(0)] \geqslant 0$, where the equality is possible for $k \neq 0$ and $\sigma \neq 0$ only if $K(0) < J(0)$, which is the case of interest for us.

Hereafter, we will focus on quantum criticality related to the in-plane ordering and hence on quantities like ω_0 and $\chi_\perp$

as functions of T and h. However, is some situations, we will present also explicit expressions for σ and/or $\chi_{\parallel}$ (near the polarized state). In the remaining cases these quantities and other ones, as the free energy, specific heat, etc., can be easily derived using known thermodynamic relations.

III. QUANTUM CRITICAL POINT AND FINITE-TEMPERATURE CRITICAL LINE FOR IN-PLANE ORDERING

For obtaining the relevant properties of our model one must solve the set of self-consistent equations (2.8), (2.12), and (2.18).

At zero temperature with $h\neq 0$, one easily finds the solution $\sigma=1$ (full polarized state) and $\omega_{\mathbf{k}}=\omega_0+S[J(0)-J(\mathbf{k})]$ with $\omega_0=h-S[J(0)-K(0)]$. Thus, the transverse susceptibility reads

$$\chi_{\perp}=\frac{2S}{h-S[J(0)-K(0)]} \tag{3.1}$$

for $h\geqslant S[J(0)-K(0)]>0$.

Equation (3.1) suggests that, for any d and S with well defined $K(0)$ and $J(0)$, a QCP exists at the value $h_c=h_c(S)=S[J(0)-K(0)]$ of the applied longitudinal field. Decreasing h to h_c, this point signals a field-induced QPT from a fully polarized state ($m=S$), without transverse ordering, to a transverse ordered phase (inaccessible in the present analysis). Moreover, we get

$$\chi_{\perp}\sim (h-h_c)^{-1}, \quad h\rightarrow h_c^{+}, \tag{3.2}$$

defining the MF exponent $\gamma_h=1$.[55] In the general case of finite temperature the transverse susceptibility and the energy gap can be usefully rewritten as

$$\chi_{\perp}=\frac{2S\sigma}{\omega_0(\sigma)}=\frac{2S\sigma}{h-\sigma h_c}, \tag{3.3}$$

and

$$\omega_0(\sigma)=h-\sigma h_c, \tag{3.4}$$

where σ must be consistently determined as a function of T and h, with $h\geqslant\sigma(T,h)h_c$ for stability conditions.

Thus, one can define a critical line in the (h,T) plane, whose existence depends on the functional structure of the exchange interactions, setting $\chi_{\perp}(T,h)=\infty$ to obtain the representation

$$\omega_0(\sigma(T,h))=h-\sigma(T,h)h_c=0 \tag{3.5}$$

or

$$\sigma(T,h)=\frac{h}{h_c}. \tag{3.6}$$

Combining Eq. (3.6) with Eq. (2.18) for σ we have the critical line equation in terms of T and h,

$$\frac{h}{h_c}=B_S(\Phi_c), \tag{3.7}$$

where the function $B_S(\Phi)$ is defined in Eq. (2.18) and

$$\Phi_c=\Phi\left(T,h;\frac{h}{h_c}\right)=\int_{1BZ}\frac{d^dk}{(2\pi)^d}\frac{1}{\exp\left\{\beta S\frac{h}{h_c}[J(0)-J(\mathbf{k})]\right\}-1}. \tag{3.8}$$

Notice that previous equations are formally true for arbitrary $J(\mathbf{k})$ and the finite-temperature critical line will exist only for suitable values of d for which integral (3.8) converges.

In principle, solving Eq. (3.7) with respect to h or T, one can obtain the full critical equation in the form $h_c(T)$ or $T_c(h)$. For instance, with $h\rightarrow h_c(T)$ the equation for $h_c(T)$ is formally given by

$$h_c(T)-h_c\sigma(T,h_c(T))=0. \tag{3.9}$$

Notice that, if $h_c(T)$ is independently known, along the critical line the reduced magnetization $\sigma(T,h_c(T))\equiv\sigma(T)$ will be simply given by $\sigma(T)=h_c(T)/h_c$ with $\sigma(0)=1$.

Starting from Eq. (3.7), under suitable convergency conditions for integral (3.8), it is easy to obtain a formal expression for the zero-field critical temperature $T_c(h=0)\equiv T_c$ where σ vanishes for a generic $J(\mathbf{k})$. Noting indeed that $\Phi_c\rightarrow\infty$ as $h\rightarrow 0$ at finite S and T, in Eq. (3.7) one has

$$\begin{aligned}B_S(\Phi_c)&\simeq\frac{S+1}{3}\frac{1}{\Phi_c}\left\{1-\frac{1}{2\Phi_c}-\frac{(2S+1)2-19}{60}\left(\frac{1}{\Phi_c}\right)^2+\cdots\right\}\\&\simeq\frac{S+1}{3}\frac{1}{\Phi_c}+O\left(\left(\frac{1}{\Phi_c}\right)^2\right),\quad \Phi_c\rightarrow\infty(h\rightarrow 0).\end{aligned} \tag{3.10}$$

Thus, in the limit $h\rightarrow 0$, Eq. (3.7) reduces to

$$\frac{h}{h_c}\simeq\frac{S+1}{3}\left[\int_{1BZ}\frac{d^dk}{(2\pi)^d}\frac{1}{\exp\left\{\beta S\frac{h}{h_c}[J(0)-J(\mathbf{k})]\right\}-1}\right]^{-1}. \tag{3.11}$$

On the other hand, with $\beta=\beta_c\neq 0$ and $h\rightarrow 0$, Eq. (3.8) provides $\Phi_c\simeq(T_c/S)(h/h_c)^{-1}\int_{1BZ}d^dk(2\pi)^{-d}/[J(0)-J(\mathbf{k})]$ and hence we have

$$T_c=\frac{S(S+1)}{3}\left[\int_{1BZ}\frac{d^dk}{(2\pi)^d}\frac{1}{J(0)-J(\mathbf{k})}\right]^{-1}. \tag{3.12}$$

For case of short-range interactions, with $X(\mathbf{k})=2X\Sigma_{\nu=1}^{d}\cos k_\nu(X=J,K>0)$, we have $X(\mathbf{k})\simeq X(0)-Xk^2$ as $k\rightarrow 0$ and one can immediately check that for $d\leqslant 2$ only the QCP $(h_c,T=0)$ exists, while a finite-temperature critical line, ending in the QCP, takes place for $d>2$, consistently with the MW theorem.[25] We do not consider here this case explicitly for arbitrary S [for $S=\frac{1}{2}$ all the results can be obtained from Ref. 38 with $K/N\rightarrow K(0)=2dK$]. Rather we focus on the case of long-range interactions which contains the short-range scenario as a particular case.

From now on, we will assume $X_{ij}=X/r_{ij}^{\alpha}(X=J,K)$ where $r_{ij}=|\mathbf{r}_i-\mathbf{r}_j|$ is the distance between the sites i and j, and J and K measure the strength of the in-plane and longitudinal couplings, respectively (the limit $\alpha\rightarrow\infty$ corresponds to the standard nearest-neighbor interaction). For this type of interactions, a sensible thermodynamic limit as $N\rightarrow\infty$ is achieved only for $\alpha>d$.[27,30,32,33] Useful representations of the Fourier transform $X(\mathbf{k})$ of X_{ij} are given in Appendix A. In the limit as $\mathbf{k}\to 0$ of interest to us, these provide to leading order in $|\mathbf{k}| = k$,[27,30,32,33]

$$X(\mathbf{k}) \simeq X(0) - X\begin{cases} A_d k^{\alpha-d}, & d<\alpha<d+2 \\ B_d k^2 \ln \dfrac{1}{k}, & \alpha = d+2 \\ C_d k^2, & \alpha > d+2, \end{cases} \tag{3.13}$$

where

$$X(0) = X\begin{cases} \dfrac{2\pi^{d/2}\Lambda_{1BZ}^{\alpha-d}}{(\alpha-d)\Gamma(d/2)}, & \alpha \neq d+2 \\ \dfrac{\pi^{d/2}\Lambda_{1BZ}^2}{\Gamma(d/2)}, & \alpha = d+2, \end{cases} \tag{3.14}$$

and

$$A_d = \frac{2^{d-\alpha}\pi^{1+d/2}}{\Gamma\left(\dfrac{\alpha}{2}\right)\Gamma\left(\dfrac{\alpha}{2}+1-\dfrac{d}{2}\right)\sin[\pi(\alpha-d)]}, \tag{3.15a}$$

$$B_d = \frac{\pi^{d/2}}{d\Gamma\left(\dfrac{d}{2}\right)}, \tag{3.15b}$$

$$C_d = \frac{\pi^{d/2}\Lambda_{1BZ}^{\alpha-(d+2)}}{2[\alpha-(d+2)]\Gamma\left(1+\dfrac{d}{2}\right)}. \tag{3.15c}$$

Here, Λ_{1BZ} is a natural wave-vector cutoff related to the definition of the first Brillouin zone, and in the present case, it is given by $1/N\Sigma_{\mathbf{k}}1=1$ or (as $N\to\infty$) $K_d\Lambda_{1BZ}^d/d=1$ with $K_d = 2^{1-d}\pi^{-d/2}/\Gamma(d/2)$. Previous results imply that the energy spectrum, to leading order in k as $k\to 0$, behaves as (we will drop the argument σ unless necessary)

$$\omega_{\mathbf{k}} \simeq \omega_0 + S\sigma J\begin{cases} A_d k^{\alpha-d}, & d<\alpha<d+2 \\ B_d k^2 \ln \dfrac{1}{k}, & \alpha = d+2 \\ C_d k^2, & \alpha > d+2, \end{cases} \tag{3.16}$$

with $\omega_0=0$ at the QCP.[56]

By inspection of Eq. (3.8), consistently with the extended MW theorem,[27] we immediately have that a finite-temperature CL, ending at the QCP, exists only for $d<\alpha<2d$ with $d\leqslant 2$ and for $\alpha>d$ with $d>2$. Beyond this range of values of α and d, i.e., for $\alpha\geqslant 2d$ and $d\leqslant 2$, only the QCP takes place.

The global scenario in the (α,d) plane is schematically plotted in Fig. 1. Here we have also distinguished the subregions of the (α,d) plane where, as we will show in the next sections, our spin model exhibits a quantum criticality characteristic of a PFM when SRIs and LRIs become effective. Within the region R_{CL}, the phase boundary is fully described by the general equation (3.7) [using in Φ_c (see Eq. (3.8)] the Fourier transform $J(\mathbf{k})$ of J_{ij} reported in Appendix A). Plots

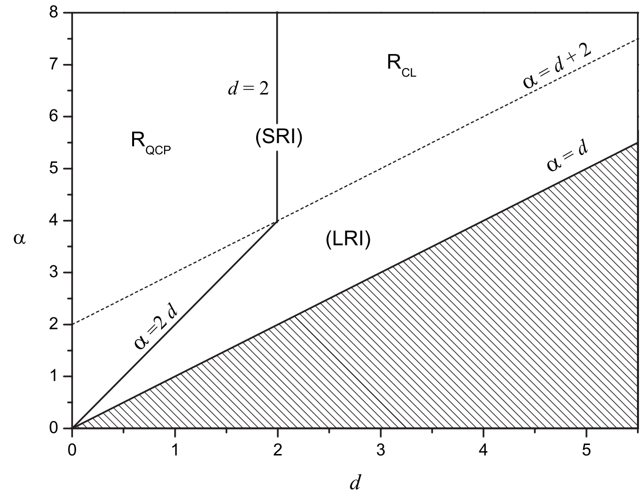

FIG. 1. Regions of the (α,d) plane where a finite-temperature critical line for the onset of transverse ordering exists (R_{CL}) and only a QCP takes place (R_{QCP}). The full lines signal borderline regimes and the dashed line ($\alpha=d+2$) separates the subregions where LRI- and SRI-behaviors occur. The shaded region with $\alpha \leqslant d$ corresponds to nonextensivity states.

in the $[h/h_c(S),T/T_c(S)]$-plane are shown in Fig. 2 for different values of S, α, and d in the region R_{CL} below the line $\alpha=d+2$. As we see a common feature is that, in the limit $S\to\infty$, all the different critical lines overlap and decrease linearly in h_c-h as $h\to h_c^-$.

No similar results are available for a useful control ground of our findings about the critical line for values of α and d within the region LRI of Fig. 1.

An estimate of the CL can be obtained through Eq. (3.7) with Φ_c evaluated assuming in the integrand expressions (3.13) for $J(0)-J(\mathbf{k})$ as $\mathbf{k}\to 0$. In the low-temperature regime, sufficiently close to the QCP, Eq. (3.7) implies $\Phi_c \ll 1$ so that one can write $h/h_c\simeq 1-\Phi_c/S$ as $T\to 0$. Thus, for any finite S to leading order in Φ_c/S the critical line equation, in the different R_{CL} subregions in Fig. 1, reduces to

(i) for $d<\alpha<d+2$ and $\alpha<2d$,

$$h-h_c+\frac{d}{\alpha-d}\frac{h_c}{S}\left(\frac{h_c}{h}\right)^{d/(\alpha-d)}\left(\frac{T}{\tau}\right)^{d/(\alpha-d)}\int_0^{(h/h_c)(\tau/T)}dx\frac{x^{d/(\alpha-d)-1}}{e^x-1} = 0, \tag{3.17}$$

(ii) for $\alpha=d+2$ and $d>2$,

$$h-h_c+\frac{d}{2}\frac{h_c}{S}\left(\frac{h_c}{h}\right)^{d/2}\left(\frac{T}{\tau}\right)^{d/2}\int_0^{(h/h_c)(\tau/T)} \times dx\frac{x^{d/2-1}}{\exp\left[x\ln\left(\dfrac{h}{h_c}\dfrac{\tau}{T}\Big/\Lambda_{1BZ}^2 x\right)\right]-1} = 0, \tag{3.18}$$

(iii) for $\alpha>d+2$ and $d>2$,

$$h-h_c+\frac{d}{2}\frac{h_c}{S}\left(\frac{h_c}{h}\right)^{d/2}\left(\frac{T}{\tau}\right)^{d/2}\int_0^{(h/h_c)(\tau/T)}dx\frac{x^{d/2-1}}{e^x-1} = 0, \tag{3.19}$$

where

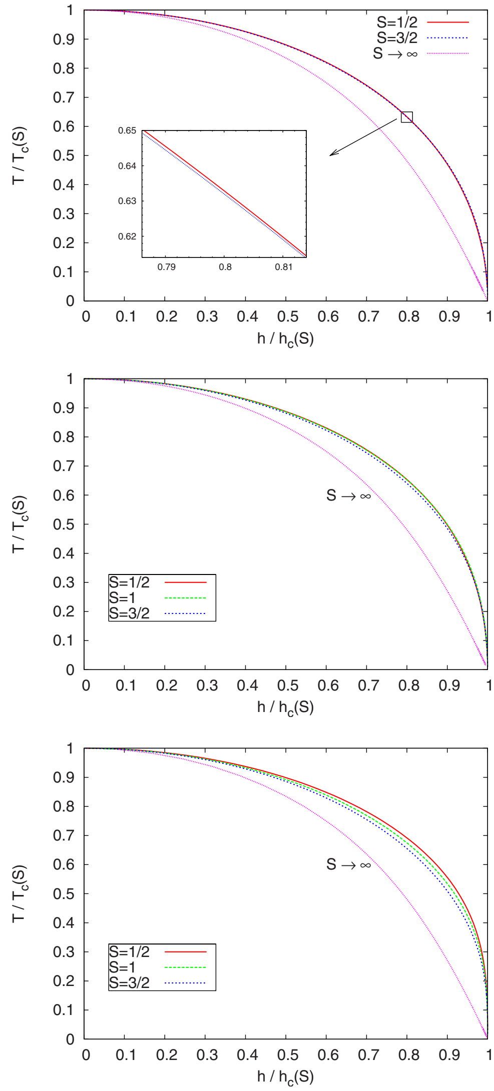

FIG. 2. (Color online) Critical lines of the PFM for some values of d and α within the region R_{CL} of the (α,d) plane, where the LRIs are expected to be effective. (a) $d=1$ and $\alpha=3/2$, here due to the closeness of the curves we also show a zoom of the lines for $S=\frac{1}{2}$ and $S=\frac{3}{2}$ in the inset; (b) $d=2$ and $\alpha=3$; and (c) $d=3$ and $\alpha=4$. As $S\to\infty$ the different critical lines overlap and, as $h\to h_c^-(S)$, decrease linearly in $h_c(S)-h$.

$$\tau = SJ\begin{cases} A_d\Lambda_{1BZ}^{\alpha-d} & (d<\alpha<d+2,\ \alpha<2d) \\ B_d\Lambda_{1BZ}^{2} & (\alpha=d+2,\ d>2) \\ C_d\Lambda_{1BZ}^{2} & (\alpha>d+2,\ d>2). \end{cases} \tag{3.20}$$

In the previous equations, the leading contribution in T as $T\to 0$ can be obtained by calculating the integrals as $\frac{\tau}{T}\gg 1$. This is explicitly performed in Appendixes B and C and Eqs. (3.17)–(3.19) provide as $T\to 0$:

(i) ($d<\alpha<d+2$ and $\alpha<2d$)

$$h-h_c+\frac{d}{\alpha-d}\frac{h_c}{S}\Gamma\left(\frac{d}{\alpha-d}\right)\zeta\left(\frac{d}{\alpha-d}\right)\left(\frac{h_c}{h}\right)^{d/(\alpha-d)}\left(\frac{T}{\tau}\right)^{d/(\alpha-d)} = 0, \tag{3.21}$$

(ii) ($\alpha=d+2$ and $d>2$)

$$h-h_c+d2^{(d-2)/2}\frac{h_c}{S}\Gamma\left(\frac{d}{2}\right)\zeta\left(\frac{d}{2}\right)\left(\frac{h_c}{h}\right)^{d/2}\left(\frac{T}{\tau}\right)^{d/2}\ln^{-d/2}\left(\frac{h}{h_c}\frac{\tau}{T}\right) = 0, \tag{3.22}$$

(iii) ($\alpha>d+2$ and $d>2$)

$$h-h_c+\frac{d}{2}\frac{h_c}{S}\Gamma\left(\frac{d}{2}\right)\zeta\left(\frac{d}{2}\right)\left(\frac{h_c}{h}\right)^{d/2}\left(\frac{T}{\tau}\right)^{d/2} = 0. \tag{3.23}$$

Notice that for one- and two-dimensional PFMs, Eq. (3.17) is true for $1<\alpha<2$ and $2<\alpha<4$, respectively.

In cases (i) and (iii), when no logarithmic corrections occur, solving for h or T provides the low-temperature representations for the critical line

$$h_c(T)\simeq h_c-\frac{\nu h_c}{S}\Gamma(\nu)\zeta(\nu)\left(\frac{T}{\tau}\right)^{\nu},\quad T\to 0, \tag{3.24}$$

and

$$T_c(h)\simeq \tau\left[\frac{\nu}{S}\Gamma(\nu)\zeta(\nu)\right]^{-1/\nu}\left(\frac{h_c-h}{h_c}\right)^{1/\nu},\quad h\to h_c^-. \tag{3.25}$$

These define the shift exponent ψ, which determines the shape of the phase boundary close to the QCP, as

$$\psi=\nu=\begin{cases} \dfrac{d}{\alpha-d}, & (d<\alpha<d+2,\ \alpha<2d) \\ \dfrac{d}{2}, & (\alpha>d+2,\ d>2). \end{cases} \tag{3.26}$$

It is worth nothing that for $d>2$ and $\alpha>d+2$, result (3.26) reproduces that obtained in Ref. 38 and in Ref. 20 (via the RG approach to the functional representation[40]) for short-range interactions with $S=1/2$.

For low-dimensional magnetic systems, we have $\psi=\frac{1}{\alpha-1}$, for $d=1$ and $1<\alpha<2$, and $\psi=\frac{2}{\alpha-2}$ for $d=2$ and $2<\alpha<4$.

In the subregion ($\alpha=d+2, d>2$) [case (ii)], very close to the QCP, one easily finds the critical line representations

$$h_c(T)\simeq h_c-d2^{(d-2)/2}\frac{h_c}{S}\Gamma(d/2)\zeta(d/2)\left(\frac{T}{\tau}\right)^{d/2}\left[\ln\left(\frac{\tau}{T}\right)\right]^{-d/2}, \tag{3.27}$$

and

$$T_c(h)\simeq \tau\frac{2}{d}\left[\frac{d2^{(d-2)/2}}{S}\Gamma(d/2)\zeta(d/2)\right]^{-2/d}\left(\frac{h_c-h}{h_c}\right)^{2/d}\times\left[\ln\left(\frac{h_c}{h_c-h}\right)\right], \tag{3.28}$$

involving logarithmic corrections.

IV. QUANTUM CRITICALITY AND CROSSOVERS

In this section we explore the low-temperature properties and crossovers for our PFM model solving the self-

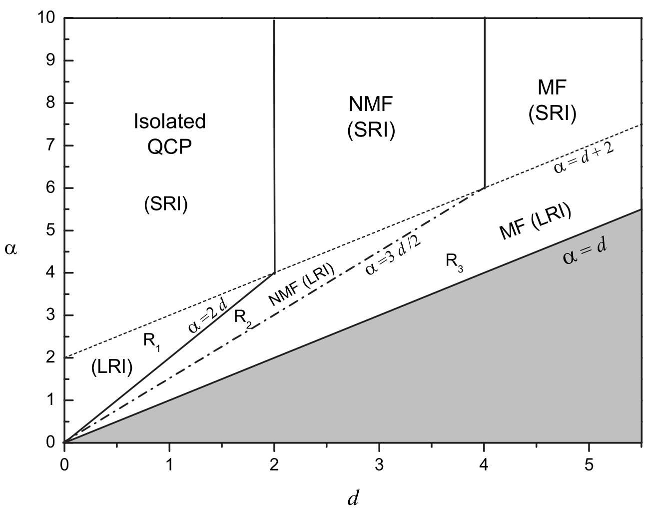

FIG. 3. Regions of the (α,d) plane where different low-temperature regimes occur close to the QCP. The related results for transverse susceptibility $\chi_\perp$ will be derived in the present section. The dashed line $\alpha=d+2$, characterized by peculiar behaviors, separates the two wide regions $\alpha<d+2$ and $\alpha>d+2$, where the LRI- and SRI-regimes take place, respectively. In the region $\alpha>d+2$ for $d\leqslant 2$ we have in-plane ordering at a critical magnetic field h_c only at zero temperature while, for $d>2$, it may occur for all values of $\alpha>4$ also at finite temperature with usual non-mean-field (NMF) and MF regimes. In the region $\alpha<d+2$, different LRI quantum critical regimes are recovered depending on the values of α and d. R_1 denotes the subregion where only the QCP exists. The line $\alpha=2d$ is a borderline between R_1 and the region $\alpha<2d$ where a finite temperature critical line, ending in the QCP, exists. In the subregion R_2, NMF behaviors are found while, in R_3, a MF regime takes place. The dashed-dot line represents the boundary between these two subregions where the system exhibits logarithmic corrections to the MF behavior.

consistent equations (2.8), (2.12), and (2.18) close to the QCP within the disordered phase where, remarkably, the problem simplifies sensibly. We will identify different subregions in the (α,d) plane characterized by distinct low-temperature regimes, as summarized in Fig. 3.

We first show that the previous self-consistent problem simplifies sensibly when one work near the QCP.

At sufficiently low-temperatures with $\sigma\simeq 1$, $h\simeq h_c$ and $\omega_0\ll 1$, Eq. (2.12) implies that, for any fixed S, one has $\Phi\ll 1$. Thus, to leading order in Φ, the self-consistent equation for σ reduces to

$$\sigma\simeq 1-\frac{\Phi(\sigma)}{S}+O(\Phi^{2S+1}(\sigma)), \tag{4.1}$$

where Φ, near criticality, can be reliably estimated from Eq. (2.12) assuming for the energy spectrum $\omega_{\mathbf{k}}$ the low-$\mathbf{k}$ expressions (3.16). We begin our analysis under condition $\alpha\neq d+2$, delaying to a separate study the most delicate borderline case $\alpha=d+2$.

A. Quantum criticality for $\alpha\neq d+2$

Near the QCP, the relevant physics in the different regions of the (α,d) plane (see Fig. 3) can be extracted from the self-consistent equation

$$\sigma\simeq 1-\frac{\nu}{S}\left(\frac{T}{\tau}\right)^{\nu}F_\nu\left(\frac{\omega_0(\sigma)}{T}\right). \tag{4.2}$$

Here, ν is given by Eq. (3.26) and

$$F_\nu(y)=\int_0^\infty dx\frac{x^{\nu-1}}{e^{x+y}-1}, \tag{4.3}$$

is the Bose-Einstein function,[57,58] whose well-known series expansions and the asymptotic behaviors of interest for us, are summarized in Appendix B.

For our purposes, it is convenient to transform Eq. (4.2) in a self-consistent equation for the energy gap $\omega_0=h-\sigma h_c\geqslant 0$ which is a very small parameter close to the QCP.

In terms of the single natural variable ω_0/T, from Eq. (4.2) we easily get

$$\frac{\omega_0}{T}\simeq\frac{g}{T}+\frac{\nu}{S}\frac{h_c}{\tau}\left(\frac{T}{\tau}\right)^{\nu-1}F_\nu\left(\frac{\omega_0}{T}\right), \tag{4.4}$$

where $g=h-h_c$.

Within the region $\alpha>d+2$, we have $\nu=d/2$ and Eq. (4.4) reduces to that for SRIs (see Ref. 38) with $K(0)=2dK$ and the factor $d/2S$ replacing K/N and d, respectively. This feature allows us to extend immediately the already known SRI quantum critical scenario found in Ref. 38 to our spin-S model when $\alpha>d+2$.

Below we focus on the most interesting region $d<\alpha<d+2$ where the long-range nature of the exchange interactions becomes effective, especially for low dimensionalities.

Within this region, $\nu=d/(\alpha-d)$ and the basic self-consistent equation for ω_0/T assumes the form

$$\frac{\omega_0}{T}\simeq\frac{g}{T}+\frac{1}{S}\frac{d}{\alpha-d}\frac{h_c}{\tau}\left(\frac{T}{\tau}\right)^{(2d-\alpha)/(\alpha-d)}F_{d/(\alpha-d)}\left(\frac{\omega_0}{T}\right), \tag{4.5}$$

which governs all the properties and crossovers close to the QCP.

A general result, valid for any $d<\alpha$, can be simply obtained for $h>h_c$ and $\omega_0/T\gg 1$. In this regime, with $F_{d/(\alpha-d)}(\omega_0/T)\simeq\Gamma(d/(\alpha-d))e^{-\omega_0/T}$, see Appendix B, we easily find for the energy gap ω_0 the asymptotic solution

$$\omega_0(T,h)\simeq(h-h_c)+\frac{1}{S}\frac{d}{\alpha-d}\Gamma\left(\frac{d}{\alpha-d}\right)h_c\left(\frac{T}{\tau}\right)^{d/(\alpha-d)}e^{-(h-h_c)/T}, \tag{4.6}$$

under condition $T\ll h-h_c$. Then, for the transverse susceptibility, we get

$$\chi_\perp(T,h)\simeq\frac{2S}{h_c}\left(\frac{h-h_c}{h_c}\right)^{-1}\left\{1-\frac{d}{S(\alpha-d)}\Gamma\left(\frac{d}{\alpha-d}\right)\times\frac{h_c}{h-h_c}\left(\frac{T}{\tau}\right)^{d/(\alpha-d)}e^{-(h-h_c)/T}\right\}, \tag{4.7}$$

which differs from the already known MF result $\chi_\perp\simeq 2S(h-h_c)^{-1}$ at $T=0$ for an exponentially small correction in temperature. Moreover, for polarized state, we find

$$\sigma(T,h) \simeq 1 - \frac{1}{S}\frac{d}{\alpha-d}\Gamma\left(\frac{d}{\alpha-d}\right)\left(\frac{T}{\tau}\right)^{d/(\alpha-d)} e^{-(h-h_c)/T}, \tag{4.8}$$

and

$$\chi_{\parallel}(T,h) \simeq \frac{d}{\alpha-d}\Gamma\left(\frac{d}{\alpha-d}\right)\tau^{-1}\left(\frac{T}{\tau}\right)^{(2d-\alpha)/(\alpha-d)} e^{-(h-h_c)/T}. \tag{4.9}$$

We now explore the low-temperature properties of our spin model in the influence domain of the QCP within the disordered phase in the different LRI subregions shown in Fig. 3.

1. Subregion R_1: $2d<\alpha<d+2$

Here, only the QCP exists and we distinguish the two asymptotic regimes $\omega_0/T \gg 1$ ($T\chi_\perp \ll 1$) and $\omega_0/T \ll 1$ ($T\chi_\perp \gg 1$).

(a) Regime $\omega_0/T \gg 1$

In this regime, Eq. (4.5) becomes

$$\frac{\omega_0}{T} \simeq \frac{g}{T} + \frac{d\Gamma\left(\frac{d}{\alpha-d}\right)}{S(\alpha-d)}\frac{h_c}{\tau}\left(\frac{T}{\tau}\right)^{(2d-\alpha)/(\alpha-d)} e^{-\omega_0/T}, \tag{4.10}$$

where $0<\frac{d}{\alpha-d}<1$ and $-1<\frac{2d-\alpha}{\alpha-d}<0$.

One easily sees that distinct asymptotic expressions for ω_0 take place depending on the sign of g and the competition between the different terms in Eq. (4.10):

(a) at $T=0$ and $h>h_c$, we have the MF result obtained before for arbitrary d and $\alpha>d$;

(b) for $h>h_c$ and $T\ll h-h_c$, the solution (4.6) is true together with the physical results (4.7)–(4.9);

(c) for $h<h_c$ and $(h_c-h)^{(\alpha-d)/d} \ll T \ll h_c-h$, to leading order in T and h_c-h, we find

$$\omega_0(T,h) \simeq T\ln\left[\frac{d\Gamma\left(\frac{d}{\alpha-d}\right)}{S(\alpha-d)}\frac{(T/\tau)^{d/(\alpha-d)}}{(h_c-h)/h_c}\right] + O\left(\frac{T^2}{h_c-h}\ln\left[\frac{T^{d/(\alpha-d)}}{h_c-h}\right]\right), \tag{4.11}$$

which provides

$$\chi_\perp \simeq 2S\frac{h}{h_c}T^{-1}\ln^{-1}\left[\frac{d\Gamma\left(\frac{d}{\alpha-d}\right)}{S(\alpha-d)}\frac{(T/\tau)^{d/(\alpha-d)}}{(h_c-h)/h_c}\right]. \tag{4.12}$$

For polarized state, a simple algebra yields

$$\sigma \simeq \frac{h}{h_c}\left\{1-\frac{T}{h}\ln\left[\frac{d\Gamma\left(\frac{d}{\alpha-d}\right)}{S(\alpha-d)}\frac{(T/\tau)^{d/(\alpha-d)}}{(h_c-h)/h_c}\right]\right\}, \tag{4.13}$$

and $\chi_\parallel \simeq \frac{S}{h_c}(1-\frac{T}{h_c-h})$; and

(d) for $T \gg |h-h_c|$, around the vertical thermodynamic path $h=h_c$ in the (h,T) plane [usually called quantum critical trajectory (QCT) Refs. 15 and 20], the suitable asymptotic expression for ω_0 is

$$\omega_0(T,h) \simeq \frac{\alpha-2d}{\alpha-d}T\ln\left(\frac{\tau}{T}\right)\left\{1+\left(\frac{\alpha-d}{\alpha-2d}\right)^2\frac{h-h_c}{T}\ln^{-2}\left(\frac{\tau}{T}\right)\right\}. \tag{4.14}$$

This provides

$$\chi_\perp \simeq \frac{2S(\alpha-d)}{\alpha-2d}\frac{h}{h_c}T^{-1}\ln^{-1}\left(\frac{\tau}{T}\right) \times\left\{1-\left(\frac{\alpha-d}{\alpha-2d}\right)^2\frac{h-h_c}{T}\ln^{-2}\left(\frac{\tau}{T}\right)\right\}, \tag{4.15}$$

with

$$\sigma \simeq \frac{h}{h_c}\left\{1-\frac{\alpha-2d}{\alpha-d}\frac{T}{h}\ln\left(\frac{\tau}{T}\right) \times\left[1+\left(\frac{\alpha-d}{\alpha-2d}\right)^2\frac{h-h_c}{T}\ln^{-2}\left(\frac{\tau}{T}\right)\right]\right\}. \tag{4.16}$$

It is worth noting that, at and sufficiently close to the QCT, one has $\omega_0 \simeq \frac{\alpha-2d}{\alpha-d}T\ln(\tau/T)$ and $\chi_\perp \simeq 2S\frac{\alpha-d}{\alpha-2d}T^{-1}\ln^{-1}(\tau/T)$.

(b) Regime $\omega_0/T \ll 1$

In this case $F_{d/(\alpha-d)}(\omega_0/T) \simeq \frac{\pi}{\sin(\pi d/(\alpha-d))}(\frac{\omega_0}{T})^{(2d-\alpha)/(\alpha-d)}$ (see Appendix B) and Eq. (4.5) assumes the form

$$\frac{\omega_0}{T} \simeq \frac{g}{T} + \frac{d}{S(\alpha-d)}\frac{\pi}{\sin\left(\frac{\pi d}{\alpha-d}\right)}\frac{h_c}{\tau}\left(\frac{T}{\tau}\right)^{(2d-\alpha)/(\alpha-d)} \times\left(\frac{\omega_0}{T}\right)^{(2d-\alpha)/(\alpha-d)}. \tag{4.17}$$

We first observe that, for $h \geq h_c$, this equation is meaningless and hence no self-consistent solution exists. In the region $h<h_c$ of the phase diagram the situation changes and a physical solution can be obtained neglecting the linear term in ω_0/T. This gives

$$\omega_0(T,h) \simeq \left[\frac{\pi d\tau^{-d/(\alpha-d)}h_c}{S(\alpha-d)\sin\left(\frac{\pi d}{\alpha-d}\right)}\right]^{(\alpha-d)/(\alpha-2d)} \times\left(\frac{T}{h_c-h}\right)^{(\alpha-d)/(\alpha-2d)}, \tag{4.18}$$

under the self-consistency condition $T \ll (h_c-h)^{(\alpha-d)/d}$ and hence we have

$$\chi_\perp(T,h) \simeq 2S\frac{h}{h_c}\left[\frac{S(\alpha-d)\sin\left(\frac{\pi d}{\alpha-d}\right)}{\pi d\tau^{-d/(\alpha-d)}h_c}\right]^{(\alpha-d)/(\alpha-2d)} \times\left(\frac{T}{h_c-h}\right)^{-(\alpha-d)/(\alpha-2d)}, \tag{4.19}$$

which signals that, for fixed $h<h_c$, the transverse susceptibility diverges as $T\to 0$ with the critical exponent $\gamma_T=\frac{\alpha-d}{\alpha-2d}$. For $d=1$ and $2<\alpha<3$ we have $\gamma_T=\frac{\alpha-1}{\alpha-2}$ with $2<\gamma_T<\infty$. In contrast, the longitudinal susceptibility remains finite as one can easily check by direct derivation, with respect to h, of the reduced magnetization

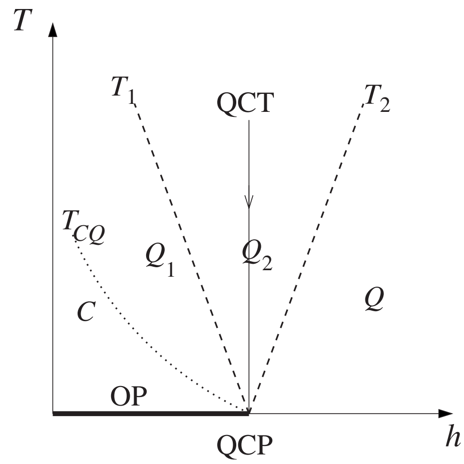

FIG. 4. Qualitative phase diagram with different low-temperature regimes close to the isolated QCP for $2d<\alpha<d+2$ with $d<2$. The line $T_{CQ}(h)$ marks the crossover between the regime $T\chi_\perp \gg 1$ (C) and $T\chi_\perp \ll 1$ (Q_1,Q_2,Q). $T_1(h)$ and $T_2(h)$ denote the crossover lines among the subregimes Q_1, Q_2, and Q increasing h. The full line OP represents the ordered phase. The meaning of notations is discussed in the text.

$$\sigma(T,h) \simeq \frac{h}{h_c}\left\{1-\left[\frac{\pi d \tau^{-d/(\alpha-d)}h_c}{S(\alpha-d)\sin\left(\frac{\pi d}{\alpha-d}\right)}\right]^{(\alpha-d)/(\alpha-2d)} \times\frac{1}{h}\left(\frac{T}{h_c-h}\right)^{(\alpha-d)/(\alpha-2d)}\right\}. \tag{4.20}$$

In summary, the near QCP phase diagram for the subregion R_1 of the (α,d) plane has the structure sketched in Fig. 4. It is substantially divided in two regions: C, where $\omega_0/T\ll 1$ $(T\chi_\perp \gg 1)$ and (Q_1,Q_2,Q), where $\omega_0/T\gg 1$ $(T\chi_\perp\ll 1)$.

The line $T_{CQ}(h)\simeq(h_c-h)^{(\alpha-d)/\alpha}$ for $h<h_c$ signals the crossovers between these two regimes. The lines $T_1(h)\simeq h_c-h$ and $T_2(h)\simeq h-h_c$, symmetric to the QCT, stand for a signature of the crossover among three sub-regimes (Q_1,Q_2,Q) characterized by different behaviors of the macroscopic quantities as functions of T and h. Within the central V-shaped region around the QCT, delimited by two crossover lines $T_1(h)$ and $T_2(h)$, the same T-behaviors as along the QCT takes place, except for small corrections in $h-h_c$.

2. Line l_1: $(\alpha=2d, d<2)$

Along this line, which is the lowest boundary of the region R_1 in the (α,d) plane (see Fig. 3), only the QCP exists and one has exactly $F_1(\omega_0/T)=\ln[1-e^{-\omega_0/T}]^{-1}$ (Appendix B). Thus, Eq. (4.5) reads

$$\frac{\omega_0}{T} \simeq \frac{g}{T}+\frac{1}{S}\left(\frac{h_c}{\tau}\right)\ln[1-e^{-\omega_0/T}]^{-1}. \tag{4.21}$$

Notice that Eq. (4.21) does not depend on d and hence it is true for the physical values $d=1$ and $\alpha=2$, $[J(r)\propto r^{-2}]$.

Focusing on asymptotic regimes close to the QCP, when $\omega_0/T\gg 1$ Eq. (4.21) has a solution only for $h>h_c$ and $T \ll h-h_c$, as expected on the ground of the previous general statements, and the relevant macroscopic quantities follow simply setting $\alpha=2d$ in Eqs. (4.6)–(4.9).

In the opposite regime $\omega_0/T\ll 1$, one can easily check that a solution takes place only for $h<h_c$ and $T\ll h_c-h$ which sounds as

$$\omega_0(T,h) \simeq Te^{-S(\tau/h_c)[(h_c-h)/T]}. \tag{4.22}$$

This provides

$$\chi_\perp(T,h) \simeq 2S\frac{h}{h_c}T^{-1}e^{S(\tau/h_c)[(h_c-h)/T]}, \tag{4.23}$$

$$\sigma(T,h) \simeq \frac{h}{h_c}\left[1-\frac{T}{h}e^{-S(\tau/h_c)[(h_c-h)/T]}\right], \tag{4.24}$$

and

$$\chi_\parallel(T,h) \simeq \frac{S}{h_c}\left[1-S\left(\frac{\tau}{h_c}\right)e^{-S(\tau/h_c)[(h_c-h)/T]}\right]. \tag{4.25}$$

Equation (4.23) shows that the transverse susceptibility diverges exponentially as $T\to 0$ for $g<0$ $(\gamma_T=\infty)$, as it happens in the region C of the phase diagram (Fig. 4) below the crossover line $T_{CQ}\simeq(h_c-h)^{(\alpha-d)/d}$ for $\alpha\to 2d^+$ [see Eq. (4.19)]. Thus, along the full line l_1, the phase diagram has a structure similar to that for the region R_1 but now (for $\alpha=2d$) the lines T_{CQ} and T_1 coincide and the intermediate regime Q_1 is absent. However, in the V-shaped region between the symmetric lines $T_1(h)\simeq h_c-h$ and $T_2(h)\simeq h-h_c$ around the QCT, a peculiar feature occurs.

Indeed, for $T\gg|h-h_c|$, a self-consistent solution of Eq. (4.21) can be shown to exist assuming $\frac{\omega_0}{T}=O(1)$. Under this condition, to leading order in $|h-h_c|/T$, we easily get

$$\omega_0(T,h) \simeq CT\left\{1+\frac{1}{C}\left[1+\frac{1}{S}\left(\frac{h_c}{\tau}\right)(e^C-1)^{-1}\right]^{-1}\frac{h-h_c}{T}\right\}. \tag{4.26}$$

where C is a positive constant determined by the equation $x+x^{S(\tau/h_c)}=1$ with $x=e^{-C}<1$. Equation (4.26) implies

$$\chi_\perp(T,h) \simeq \frac{2S}{C}\frac{h}{h_c}T^{-1} \times\left\{1-\frac{1}{C}\left[1+\frac{1}{S}\left(\frac{h_c}{\tau}\right)(e^C-1)^{-1}\right]^{-1}\frac{h-h_c}{T}\right\}. \tag{4.27}$$

In particular, for $h=h_c$ as $T\to 0$ along the QCT, we have $\omega_0\simeq CT$ and $\chi_\perp\simeq\frac{2S}{C}T^{-1}$ defining the MF-like exponent $\gamma_T=1$. Equation (4.27) shows also that, sufficiently close to the QCT, the same T-dependent behavior is expected except for (with) small corrections in $h-h_c$.

3. Subregion R_2: $\{\frac{3}{2}d<\alpha<2d, \alpha<d+2,\}$

In this subregion of the (α,d) plane, a critical line exists ending in the QCP and also here one is forced to solve Eq. (4.5) in asymptotic regimes within the disordered phase for obtaining explicit results.

For $\omega_0/T \gg 1$, as already shown for any $d<\alpha$, the energy gap and other related thermodynamic quantities are again given by Eqs. (4.6)–(4.9) for $h>h_c$ and $T \ll h-h_c$, providing leading MF behaviors as at $T=0$ (Q region), with exponentially small corrections in the temperature.

To explore the most interesting regime $\omega_0/T \ll 1$ within the region of the phase diagram delimited by the crossover line $T_2(h) \simeq h-h_c$ and the critical line, we conveniently rewrite the basic equation (4.5) as

$$\frac{\omega_0}{T} \simeq \frac{g(T)}{T} + \frac{1}{S}\frac{d}{(\alpha-d)}\left(\frac{h_c}{\tau}\right)\left(\frac{T}{\tau}\right)^{(2d-\alpha)/(\alpha-d)} G_{d/(\alpha-d)}\left(\frac{\omega_0}{T}\right), \tag{4.28}$$

where now $1<\frac{d}{\alpha-d}<2$ and $0<\frac{2d-\alpha}{\alpha-d}<1$. Here, $g(T)=h-h_c(T)\geq 0$ measures the horizontal distance from the phase boundary, $h_c(T)$ is given by Eq. (3.24) with $\psi=\frac{d}{\alpha-d}$, and $G_{d/(\alpha-d)}(\omega_0/T)=F_{d/(\alpha-d)}(\omega_0/T)-F_{d/(\alpha-d)}(0)$, where $F_{d/(\alpha-d)}(0)=\Gamma(\frac{d}{\alpha-d})\zeta(\frac{d}{\alpha-d})$.

Notice that in terms of $T_c(h)$ [see Eqs. (3.25) and (3.26)], for $h\leq h_c$ one can write $g(T)\simeq \frac{1}{S}\frac{d}{\alpha-d}F_{d/(\alpha-d)}(0)h_c\,[(\frac{T}{\tau})^{d/(\alpha-d)}-(\frac{T_c(h)}{\tau})^{d/(\alpha-d)}]$, measuring now the vertical distance from the critical line for fixed $h\leq h_c$ and $T\geq T_c(h)$.

In the regime of interest, $G_{d/(\alpha-d)}(\omega_0/T)\simeq -\frac{\pi}{|\sin(\frac{\pi d}{\alpha-d})|}(\frac{\omega_0}{T})^{(2d-\alpha)/(\alpha-d)}$ (see Appendix B) and Eq. (4.28) reduces to

$$\frac{\omega_0}{T} \simeq \frac{g(T)}{T} - \frac{1}{S}\frac{d}{(\alpha-d)}\frac{\pi(h_c)/\tau}{\left|\sin\left(\frac{\pi d}{\alpha-d}\right)\right|}\left(\frac{T}{\tau}\right)^{(2d-\alpha)/(\alpha-d)} \times\left(\frac{\omega_0}{T}\right)^{(2d-\alpha)/(\alpha-d)}. \tag{4.29}$$

Asymptotic solutions of this equation can be easily obtained in the two subregimes $\frac{\omega_0}{T}\ll\Omega$ and $\frac{\omega_0}{T}\gg\Omega$ where

$$\Omega = \frac{1}{S}\frac{\pi d/(\alpha-d)}{\left|\sin\left(\frac{\pi d}{\alpha-d)}\right)\right|}\left(\frac{h_c}{\tau}\right)\left(\frac{T}{\tau}\right)^{(2d-\alpha)/(\alpha-d)}\left(\frac{\omega_0}{T}\right)^{(2d-\alpha)/(\alpha-d)}. \tag{4.30}$$

Approaching the phase boundary as $h\to h_c^+(T)$ along horizontal trajectories, when $\omega_0/T\ll\Omega$, we have

$$\omega_0(T,h) \simeq \tau\left[\frac{\left|\sin\left(\frac{\pi d}{\alpha-d}\right)\right| S}{\pi d/(\alpha-d)}\right]^{(\alpha-d)/(2d-\alpha)}\left(\frac{\tau}{T}\right)^{(\alpha-d)/(2d-\alpha)} \times\left(\frac{h_c(T)}{h_c}\right)^{(\alpha-d)/(2d-\alpha)}\left(\frac{h-h_c(T)}{h_c(T)}\right)^{(\alpha-d)/(2d-\alpha)}. \tag{4.31}$$

Then, focusing on transverse susceptibility, we get

$$\chi_\perp(T,h) \simeq \frac{2S}{\tau}\frac{h}{h_c}\left[\frac{\left|\sin\left(\frac{\pi d}{\alpha-d}\right)\right| S}{\pi d/(\alpha-d)}\right]^{-(\alpha-d)/(2d-\alpha)}\left(\frac{T}{\tau}\right)^{(\alpha-d)/(2d-\alpha)} \times\left(\frac{h_c(T)}{h_c}\right)^{-(\alpha-d)/(2d-\alpha)}\left(\frac{h-h_c(T)}{h_c(T)}\right)^{-(\alpha-d)/(2d-\alpha)}, \tag{4.32}$$

which defines the critical exponent $\gamma_h=\frac{\alpha-d}{2d-\alpha}>1$.

According to the current terminology[15,20,56] we call this subregime as the classical critical (CC) regime (in our near QCP scenario).

For $\omega_0/T\gg\Omega$, we have trivially $\omega_0(T,h)\simeq h-h_c(T)$ which defines the MF exponent $\gamma_h=1$, as at $T=0$.

The crossover between previous asymptotic behaviors is signaled by $\omega_0/T\simeq\Omega$ which allows to define the Ginzburg-like line equation for horizontal trajectories

$$h_{Gi}(T) \simeq h_c(T) + h_c\left[\frac{\pi d/(\alpha-d)}{S\left|\sin\left(\frac{\pi d}{\alpha-d}\right)\right|}\right]^{(\alpha-d)/(2\alpha-3d)} \times\left(\frac{h_c}{\tau}\right)^{(2d-\alpha)/(2\alpha-3d)}\left(\frac{T}{\tau}\right)^{(\alpha-d)/(2\alpha-3d)}, \tag{4.33}$$

merging with the critical line at the QCP as $T\to 0$.

For vertical trajectories, with $T_c(h)$ finite at fixed $h<h_c$ and $T^{d/(\alpha-d)}-T_c(h)^{d/(\alpha-d)}\simeq\frac{d}{\alpha-d}T_c(h)^{(2d-\alpha)/(\alpha-d)}(T-T_c(h))$ as $T\to T_c^+(h)$, Eq. (4.29) yields

$$\omega_0(T,h) \simeq \begin{cases} \left[\frac{d\left|\sin\left(\frac{\pi d}{\alpha-d}\right)\right|}{(\alpha-d)\pi}F_{d/(\alpha-d)}(0)\right]^{(\alpha-d)/(2d-\alpha)} T_c(h)\left(\frac{T-T_c(h)}{T_c(h)}\right)^{(\alpha-d)/(2d-\alpha)}, & \frac{\omega_0}{T}\ll\Omega, \\ \frac{1}{S}\left(\frac{d}{\alpha-d}\right)^2 F_{d/(\alpha-d)}(0)h_c\left(\frac{T_c(h)}{\tau}\right)^{d/(\alpha-d)}\frac{T-T_c(h)}{T_c(h)}, & \frac{\omega_0}{T}\gg\Omega. \end{cases} \tag{4.34}$$

This provides, for $h<h_c$ and $T\to T_c^+(h)$,

$$\chi_\perp(T,h) \simeq \begin{cases} 2S\dfrac{h}{h_c}\left[\dfrac{(\alpha-d)\pi}{d\left|\sin\left(\dfrac{\pi d}{\alpha-d}\right)\right|F_{d/(\alpha-d)}(0)}\right]^{(\alpha-d)/(2d-\alpha)} T_c^{-1}(h)\left(\dfrac{T-T_c(h)}{T_c(h)}\right)^{-(\alpha-d)/(2d-\alpha)}, & \dfrac{\omega_0}{T}\ll\Omega \\ 2S^2\dfrac{h}{h_c^2}\left(\dfrac{\alpha-d}{d}\right)^2 F^{-1}_{d/(\alpha-d)}(0)\left(\dfrac{T_c(h)}{\tau}\right)^{-d/(\alpha-d)}\left(\dfrac{T-T_c(h)}{T_c(h)}\right)^{-1}, & \dfrac{\omega_0}{T}\gg\Omega \end{cases} \tag{4.35}$$

which defines a critical exponent $\gamma_T \equiv \gamma_h$.

The crossover between previous two asymptotic regimes is signaled by $\omega_0/T \simeq \Omega$ which allows to estimate the conventional (for vertical trajectories) Ginzburg line equation for $h < h_c$ as

$$T_{Gi}(h) \simeq T_c(h) + \tau S\left(\frac{\alpha-d}{d}\right)^2 F^{-1}_{d/(\alpha-d)}(0) \times\left[\frac{\pi d/(\alpha-d)}{S\left|\sin\left(\dfrac{\pi d}{\alpha-d}\right)\right|}\right]^{(\alpha-d)/(2\alpha-3d)}\left(\frac{h_c}{\tau}\right)^{(2d-\alpha)/(2\alpha-3d)} \times\left(\frac{T_c(h)}{\tau}\right)^{\phi}, \tag{4.36}$$

where

$$\phi = \tfrac{\alpha-d}{2\alpha-3d} - \psi + 1 = \tfrac{3\alpha^2-9\alpha d+7d^2}{(2\alpha-3d)(\alpha-d)} > 0.$$

Notice that, as $T_c(h)$, also $T_{Gi}(h)\to 0$ as $h\to h_c^-$.

The behavior of energy gap along the QCT, i.e., the so-called "quantum critical (QC)" regime decreasing T at $h = h_c$, has a particular experimental interest. Since here $T_c(h_c)=0$, we have $g(T) \simeq \frac{dh_c}{S(\alpha-d)}F_{d/(\alpha-d)}(0)(\frac{T}{\tau})^{d/(\alpha-d)}$ and the self-consistent equation for the energy gap becomes

$$\frac{\omega_0}{T} \simeq \frac{d}{S(\alpha-d)}F_{d/(\alpha-d)}(0)\left(\frac{h_c}{\tau}\right)\left(\frac{T}{\tau}\right)^{(2d-\alpha)/(\alpha-d)} \times\left\{1 - \frac{\pi}{F_{d/(\alpha-d)}(0)\left|\sin\left(\dfrac{\pi d}{\alpha-d}\right)\right|}\left(\frac{\omega_0}{T}\right)^{(2d-\alpha)/(\alpha-d)}\right\}. \tag{4.37}$$

Sufficiently close the QCP with $\omega_0/T \ll 1$, Eq. (4.37) has the asymptotic solution

$$\omega_0(T,h_c) \simeq \frac{dh_c}{S(\alpha-d)}F_{d/(\alpha-d)}(0)\left(\frac{T}{\tau}\right)^{d/(\alpha-d)}, \tag{4.38}$$

and hence we find

$$\chi_\perp(T,h_c) \simeq 2\frac{S^2}{h_c}\frac{\alpha-d}{dF_{d/(\alpha-d)}(0)}\left(\frac{T}{\tau}\right)^{-d/(\alpha-d)}, \tag{4.39}$$

which defines the critical exponent $\gamma_T = \frac{d}{\alpha-d}$ (with $1 < \gamma_T < 2$), which coincides with the shift exponent ψ [see Eq. (3.26)].

We now investigate the remaining region of the phase diagram for $h > h_c$ and $T \gg T_2(h) \simeq h - h_c$.

By inspection of Eq. (4.29), it is easy to check that, when $T \gg T_1(h) \simeq \tau[\frac{S(\alpha-d)}{dh_c F_{d/(\alpha-d)}(0)}]^{(\alpha-d)/d}(h-h_c)^{(\alpha-d)/d}$, the energy gap, the transverse susceptibility and other related macroscopic quantities behave as along the QCT. In the opposite regime [called here quantum intermediate (QI) regime] with $T_2(h) \ll T \ll T_1(h)$, $\chi_\perp$ behaves as $(h-h_c)^{-1}$ with small power-law corrections in temperature. It is worth noting that, in the previous scenario, the crossover line $T_1(h)$ is symmetric to the critical line [see Eq. (3.25)] with respect to the QCT.

In conclusion, consistently with the QCP scenario emerging from RG treatments,[15,20] the previous findings suggest that, within the region R_2 of the (α,d) plane, where the LRI's play an effective role, the phase diagram around the QCP has the qualitative aspect sketched in Fig. 5.

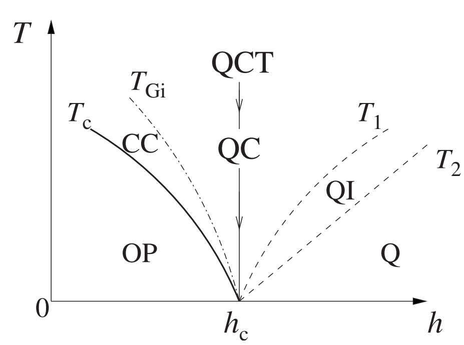

FIG. 5. Qualitative structure of the phase diagram close to the QCP for (α,d) within the subregion R_2 of Fig. 3 where LRI's are effective. Here, T_c denotes the finite-temperature critical line and T_{Gi} marks the crossovers between the MF and the CC regimes within the disordered phase. The remaining lines signal crossovers between regions where different low temperature regimes take place: Q stands for the quantum regime with $T\chi_\perp \ll 1$, $\chi_\perp$ denoting the easy-plane susceptibility, with an exponentially small correction to the $(T=0)$-MF behavior in $h-h_c$; QC denotes the quantum critical regime with $T\chi_\perp \gg 1$ and a behavior which essentially coincides with that along the QCT. QI is a quantum intermediate regime with $T\chi_\perp \gg 1$, which differs from that in the Q-regime only for a power-law small correction.

4. Line l_2: $(\alpha=\frac{3}{2}d, d<4)$

This line separates the two subregions (2) and (3) in the (α,d) plane and lies in the region where a finite-temperature critical line exists. Here $\frac{d}{\alpha-d}=2$ and Eq. (4.5) becomes

$$\frac{\omega_0}{T} \simeq \frac{g}{T} + \frac{2}{S}\left(\frac{h_c}{\tau}\right)\left(\frac{T}{\tau}\right)F_2\left(\frac{\omega_0}{T}\right). \tag{4.40}$$

Notice that, as along l_1, the dimensionality $d<4$ does not enter Eq. (4.40) explicitly so that all the related findings are true for $d=1,2,3$ with $\alpha=\frac{3}{2},3,\frac{9}{2}$ respectively.

In the quantum regime $\frac{\omega_0}{T}\gg 1$, $F_2(\frac{\omega_0}{T})\simeq e^{-\omega_0/T}$ and the relevant properties for $h>h_c$ and $T\ll h-h_c$ can be derived by Eqs. (4.6)–(4.9) with $\frac{d}{\alpha-d}=2$.

For $\frac{\omega_0}{T}\ll 1$, it is convenient to use Eq. (4.28) where now (see Appendix B) $G_2(\frac{\omega_0}{T})\simeq -\frac{\omega_0}{T}\ln(\frac{1}{\omega_0/T})$ so that the appropriate equation for ω_0/T assumes the form

$$\frac{\omega_0}{T} \simeq \frac{g(T)}{T} - \frac{2}{S}\left(\frac{h_c}{\tau}\right)\left(\frac{T}{\tau}\right)\frac{\omega_0}{T}\ln\left(\frac{1}{\omega_0/T}\right), \qquad (4.41)$$

to be compared with Eq. (4.29). Here $g(T)\simeq(h-h_c)+\frac{\pi^2}{3S}h_c(\frac{T}{\tau})^2$, with $h_c(T)\simeq h_c-\frac{\pi^2}{3S}h_c(\frac{T}{\tau})^2(\psi=2)$ or, in terms of $T_c(h)\simeq\tau(\frac{3S}{\pi^2})^{1/2}(\frac{h_c-h}{h_c})^{1/2}$ for $h\leq h_c$ [see Eqs. (3.24) and (3.25)], $g(T)\simeq\frac{\pi^2}{3S}h_c[(\frac{T}{\tau})^2-(\frac{T_c(h)}{\tau})^2]$.

With these ingredients, one can extract the relevant physical information following the procedure used in *C*. for the subregion R_2 by replacing the quantity Ω [Eq. (4.30)] with the second term in the right-hand side of Eq. (4.41).

We present here the main asymptotic results for the energy gap and the transverse susceptibility which may be of interest for low-dimensional PFMs with $\alpha=\frac{3}{2}d$ in the power-law LRIs.

Focusing on vertical trajectories within the region of the phase diagram delimited by the crossover line $T_2(h)\simeq h-h_c$ and the critical line, the following scenario emerges from Eq. (4.41).

For $h<h_c$ and $T\to T_c^+(h)$, sufficiently close to the critical line in the regime $\omega_0/T\ll\Omega$, from Eq. (4.41) we easily have

$$\omega_0(T,h) \simeq \frac{\pi^2}{3}T_c(h)\left[\frac{T-T_c(h)}{T_c(h)}\right]\ln^{-1}\left[\frac{T_c(h)}{T-T_c(h)}\right], \qquad (4.42)$$

and hence

$$\chi_\perp(T,h) \simeq 2S\frac{h}{h_c}\frac{3}{\pi^2}\frac{1}{T_c(h)}\left[\frac{T-T_c(h)}{T_c(h)}\right]^{-1}\ln\left[\frac{T_c(h)}{T-T_c(h)}\right]. \qquad (4.43)$$

In the opposite regime $\omega_0/T\gg\Omega$, MF results take place with

$$\omega_0(T,h) \simeq \frac{2\pi^2}{3S}h_c\left(\frac{T_c(h)}{\tau}\right)^2\left[\frac{T-T_c(h)}{T_c(h)}\right], \qquad (4.44)$$

and

$$\chi_\perp(T,h) \simeq \frac{3S^2}{\pi^2}\frac{h}{h_c^2}\left(\frac{\tau}{T_c(h)}\right)^2\left[\frac{T-T_c(h)}{T_c(h)}\right]^{-1}. \qquad (4.45)$$

The signature of crossover between the previous asymptotic regimes is given by $\omega_0/T\simeq\Omega$ which defines the Ginzburg-like line

$$T_{Gi}(h) \simeq T_c(h) + \tau\left[\frac{2\pi^2}{3S}\left(\frac{h_c}{\tau}\right)\right]^{-1} e^{-(S/2)(\tau/h_c)[\tau/T_c(h)]}. \qquad (4.46)$$

Along the QCT, Eq. (4.41) becomes

$$\frac{\omega_0}{T} \simeq \frac{\pi^2}{3S}\left(\frac{h_c}{\tau}\right)\left(\frac{T}{\tau}\right)\left\{1-\frac{6}{\pi^2}\left(\frac{\omega_0}{T}\right)\ln\left(\frac{1}{\omega_0/T}\right)\right\}, \qquad (4.47)$$

which provides by iteration, to leading orders in $\omega_0/T\ll 1$,

$$\omega_0(T,h) \simeq \frac{\pi^2}{3S}h_c\left(\frac{T}{\tau}\right)^2\left\{1-\frac{2}{S}\left(\frac{h_c}{\tau}\right)\left(\frac{T}{\tau}\right)\ln\left[\frac{1}{\frac{\pi^2}{3S}\left(\frac{h_c}{\tau}\right)\frac{T}{\tau}}\right]\right\}. \qquad (4.48)$$

Then, decreasing T at fixed $h=h_c$, $\chi_\perp$ behaves as

$$\chi_\perp \simeq 2S\left[\frac{3S}{\pi^2 h_c}\right]\left(\frac{T}{\tau}\right)^{-2}\left\{1+\frac{2}{S}\left(\frac{h_c}{\tau}\right)\left(\frac{T}{\tau}\right)\ln\left[\frac{3S}{\pi^2\left(\frac{h_c}{\tau}\right)\frac{T}{\tau}}\right]\right\} \sim T^{-2}, \qquad (4.49)$$

which defines the critical exponent $\gamma_T\equiv\psi=2$.

We now explore the region of the phase diagram for $h>h_c$ and $T\gg T_2(h)\simeq h-h_c$. Equation (4.41) shows that, in the regime $T\gg T_1(h)\simeq\tau(\frac{3S}{\pi^2})^{1/2}(\frac{h-h_c}{h_c})^{1/2}$, the system exhibits the same behavior (4.48) and (4.49) as along the QCT. In the opposite regime one easily finds $\omega_0\simeq(h-h_c)+\frac{\pi^2}{3S}h_c(\frac{T}{\tau})^2+O((\frac{T}{\tau})^2\frac{h-h_c}{T}\ln[\frac{1}{(h-h_c)/T}])$ so that the leading behavior in terms of $h-h_c$ is essentially the same as at $T=0$ (QMF behavior) except for $O(T^2)$ corrections.

Previous results suggest that along the line l_2 in the (α,d) plane, the phase diagram in the vicinity of the QCP has the same structure of that shown in Fig. 5, the Ginzburg-like line $T_{Gi}(h)$ signaling now the crossover from a genuine MF regime to a MF-like one with logarithmic corrections, decreasing T to $T_c(h)$.

5. Subregion R_3: $\{d<\alpha<\frac{3}{2}d, \alpha<d+2\}$

Also in this subregion of the (α,d) plane a CL ending in the QCP exists and, in the quantum regime $\omega_0/T\gg 1$, solutions (4.6)–(4.9) are again true below the line $T_2(h)\simeq h-h_c$ for $h>h_c$. For $\omega_0/T\ll 1$, within the region delimited by $T_c(h)$ and the crossover line $T_2(h)$, using the appropriate asymptotic expression for $F_{d/(\alpha-d)}(\omega_0/T)$ (see Appendix B), Eq. (4.28) reduces to

$$\frac{\omega_0}{T} \simeq \frac{g(T)}{T} - \frac{A(\alpha,d)}{S}\left(\frac{h_c}{\tau}\right)\left(\frac{T}{\tau}\right)^{(2d-\alpha)/(\alpha-d)}\left(\frac{\omega_0}{T}\right), \qquad (4.50)$$

where $A(\alpha,d)=\frac{d(2d-\alpha)}{(\alpha-d)^2}\Gamma(\frac{2d-\alpha}{\alpha-d})\zeta(\frac{2d-\alpha}{\alpha-d})$, $\frac{d}{\alpha-d}>2$ and $\frac{2d-\alpha}{\alpha-d}>1$.

Equation (4.50) provides

$$\omega_0 \simeq g(T)\left\{1 - \frac{A(\alpha,d)}{S}\left(\frac{h_c}{\tau}\right)\left(\frac{T}{\tau}\right)^{(2d-\alpha)/(\alpha-d)}\right\}. \quad (4.51)$$

So, for horizontal trajectories with $g(T)=h-h_c(T)\to 0^+$ and $T/\tau \ll 1$ we have only the MF regime with $\omega_0 \simeq g(T)$ and $\chi_\perp \sim [g(T)]^{-1}$.

For vertical trajectories, by inspection of Eq. (4.50) it is immediate to check that all the results found in the region R_2 of the (α,d) plane are formally valid except for the CC region in the phase diagram which is now absent. In particular, we find again the line $T_1(h)$, which marks the crossover between the two low-temperature regimes denoted by QC and QI in Fig. 5, and the critical exponent $\gamma_T \equiv \psi = \frac{d}{\alpha-d}$ along the QCT, where now $\gamma_T > 2$.

In conclusion, for the subregion R_3 the phase diagram preserves, qualitatively, the structure of that in Fig. 5 for the subregion R_2, but now the Ginzburg crossover regime is absent.

B. Quantum criticality along the line $\alpha = d+2$

Along the line $\alpha=d+2$, which separates the two wide regions in the (α,d) plane when the LRI's are effective ($\alpha < d+2$) and ineffective ($\alpha > d+2$) (see Fig. 3), we must carefully analyze the appropriate self-consistent equation [see Eqs. (2.11), (3.16), and (4.1)] close to the QCP.

$$\sigma \simeq 1 - \frac{d}{2S}\left(\frac{T}{\tau}\right)^{d/2} H_{d/2}\left(\frac{\tau}{T},\frac{\omega_0}{T}\right), \quad (4.52)$$

[to be compared with Eq. (4.2)], where

$$H_{d/2}\left(\frac{\tau}{T},\frac{\omega_0}{T}\right) = \int_0^{\tau/T} dx \frac{x^{\nu-1}}{e^{(\omega_0/T)+(1/2)x\,\ln\left(\frac{\tau}{T}\frac{1}{\Lambda_{1BZ}^2 x}\right)}-1}. \quad (4.53)$$

In terms of the single variable ω_0/T, with $\omega_0 = h-\sigma h_c \geq 0$, Eq. (4.52) reduces to

$$\frac{\omega_0}{T} \simeq \frac{g}{T} + \frac{d}{2S}\left(\frac{h_c}{\tau}\right)\left(\frac{T}{\tau}\right)^{(d-2)/2} H_{d/2}\left(\frac{\tau}{T},\frac{\omega_0}{T}\right), \quad (4.54)$$

where $g=h-h_c$ and $T/\tau \ll 1$.

Also in this case one is forced to solve Eq. (4.54) in asymptotic regimes as for Eq. (4.5). This can be performed after a careful analysis of the asymptotic expansions of the function $H_\nu(\delta,y)$ for different values of $\nu=d/2$, with $\delta = \tau/T \gg 1$ and $y=\omega_0/T$. The essential results are reported in Appendix C.

We first consider, as for $\alpha \neq d+2$, the regime for $h > h_c$ and $\omega_0/T \gg 1$, where $H_{d/2}(\frac{\tau}{T},\frac{\omega_0}{T}) \simeq f_{\frac{d}{2}}(\frac{\tau}{T})e^{-\omega_0/T}$ and $f_{d/2}(\frac{\tau}{T}) = \int_0^{\tau/T} dx x^{d/2-1} e^{-(1/2)x\,\ln[(\tau/T)(1/\Lambda_{1BZ}^2 x)]} \simeq 2^{d/2}\Gamma(d/2)\ln^{-d/2}(\frac{\tau}{T})$ to leading order in $\frac{\tau}{T} \gg 1$. By iteration, from Eq. (4.54) we easily find the asymptotic solution for the energy gap

$$\omega_0(T,h) \simeq (h-h_c) + \frac{dh_c}{S}2^{(d-2)/2}\Gamma(d/2) \times \left(\frac{T}{\tau}\right)^{d/2} \ln^{-d/2}\left(\frac{\tau}{T}\right) e^{-(h-h_c)/T}, \quad (4.55)$$

which is true for $T \ll h-h_c$ and any value of d. For the transverse susceptibility one has

$$\chi_\perp(T,h) \simeq \frac{2S}{h_c}\left(\frac{h-h_c}{h_c}\right)^{-1}\left\{1 - \frac{d}{S}2^{(d-2)/2}\Gamma(d/2)\frac{h_c}{h-h_c}\left(\frac{T}{\tau}\right)^{d/2} \times \ln^{-d/2}(\tau/T)e^{-(h-h_c)/T}\right\}, \quad (4.56)$$

and related expressions for $\sigma(T,h)$ and $\chi_\parallel(T,h)$ can be immediately obtained. At $T=0$, we have $\omega_0 \simeq h-h_c$ and $\chi_\perp \simeq 2S(h-h_c)^{-1}$.

We now determine the low-temperature properties of our PFM for different ranges of values of d along the different upper boundaries of the subregions R_1-R_3 considered before. The procedure is formally the same as that used in subsection IV A with the asymptotic expansions for $F_{d/(\alpha-d)}(\omega_0/T)$ replaced by those for $H_{d/2}(\frac{\tau}{T},\frac{\omega_0}{T})$.

(i) $0 < d < 2 (2 < \alpha < 4)$

Here only the QCP exists and we distinguish again the asymptotic regimes $\omega_0/T \gg 1 (T\chi_\perp \ll 1)$ and $\omega_0/T \ll 1$ $(T\chi_\perp \gg 1)$.

(a) Regime $\omega_0/T \gg 1$

By considering all the possible signs for g, Eq. (4.54) reduces to

$$\frac{\omega_0}{T} \simeq \frac{g}{T} + \frac{d}{S}2^{(d-2)/2}\Gamma(d/2)\left(\frac{h_c}{\tau}\right)\left(\frac{T}{\tau}\right)^{(d-2)/2} \ln^{-d/2}\left(\frac{\tau}{T}\right)e^{-\omega_0/T}, \quad (4.57)$$

with $-1 < \frac{d-2}{2} < 0$.

The solution for $h > h_c$ and $T \ll h-h_c$ has been already obtained before [see Eq. (4.55)].

We consider now the region on the left of the crossover line $T_2(h) \simeq h-h_c$.

For $h < h_c$ and $(h_c-h)^{2/d} \ll T \ll h_c-h$ we have the asymptotic solution

$$\omega_0(T,h) \simeq T\ln\left[\frac{d}{S}2^{(d-2)/2}\Gamma(d/2)\frac{(T/\tau)^{d/2}}{(h_c-h)/h_c}\ln^{-d/2}\left(\frac{\tau}{T}\right)\right] \quad (4.58)$$

so that

$$\chi_\perp(T,h) \simeq 2S\frac{h}{h_c}T^{-1}\ln^{-1}\left[\frac{d}{S}\frac{2^{(d-2)/2}\Gamma(d/2)(T/\tau)^{d/2}}{(h_c-h)/h_c}\ln^{-d/2}\left(\frac{\tau}{T}\right)\right]. \quad (4.59)$$

Around and at the line $h=h_c$ for $T \gg |h-h_c|$, the suitable solution for the energy gap is

$$\omega_0(T,h) \simeq \frac{2-d}{2} T \ln\left(\frac{\tau}{T}\right)\left\{1 + \left(\frac{2}{2-d}\right)^2 \frac{h-h_c}{T} \ln^{-2}\left(\frac{\tau}{T}\right)\right\}, \tag{4.60}$$

which provides for $\chi_\perp$,

$$\chi_\perp(T,h) \simeq \frac{4S}{2-d}\frac{h}{h_c} T^{-1} \ln^{-1}\left(\frac{\tau}{T}\right) \times\left\{1 - \left(\frac{2}{2-d}\right)^2 \frac{h-h_c}{T} \ln^{-2}\left(\frac{\tau}{T}\right)\right\}. \tag{4.61}$$

(b) Regime $\omega_0/T \ll 1$

Under this condition one has $H_{d/2}(\frac{\tau}{T},\frac{\omega_0}{T}) \simeq \frac{2^{d/2}\pi}{\sin(\frac{\pi d}{2})}(\frac{\omega_0}{T})^{(d-2)/2} \ln^{-d/2}(\frac{\tau}{\omega_0})$ and hence Eq. (4.54) becomes

$$\frac{\omega_0}{T} \simeq \frac{g}{T} + \frac{d}{S}\frac{2^{(d-2)/2}\pi}{\sin\left(\frac{\pi d}{2}\right)}\left(\frac{h_c}{\tau}\right)\left(\frac{\omega_0}{\tau}\right)^{(d-2)/2} \ln^{-d/2}\left(\frac{\tau}{\omega_0}\right). \tag{4.62}$$

A self-consistent solution exists only for $h < h_c$ and $T \ll h-h_c$ and one easily has

$$\omega_0(T,h) \simeq \tau\left[\frac{d}{S}\frac{2^{(d-2)/2}\pi}{\sin\left(\frac{\pi d}{2}\right)}\left(\frac{h_c}{\tau}\right)\right]^{2/(2-d)}\left(\frac{2}{2-d}\right)^{d/(2-d)} \times\left(\frac{T}{h_c-h}\right)^{2/(2-d)} \ln^{d/(2-d)}\left(\frac{h_c-h}{T}\right), \tag{4.63}$$

which provides

$$\chi_\perp(T,h) \simeq 2S\frac{h}{h_c}\left[\frac{S}{d}\frac{\sin(\pi d/2)}{2^{(d-2)/2}\pi}\left(\frac{\tau}{h_c}\right)\right]^{2/(2-d)}\left(\frac{2-d}{2}\right)^{d/(2-d)} \times\left(\frac{T}{h_c-h}\right)^{-2/(2-d)} \ln^{-d/(2-d)}\left(\frac{h_c-h}{T}\right) \sim \left(\frac{T}{h_c-h}\right)^{-2/(2-d)} \ln^{-d/(2-d)}\left(\frac{h_c-h}{T}\right). \tag{4.64}$$

Expressions for σ and $\chi_\parallel$ follow immediately from previous general relations.

(ii) $d=2$ ($\alpha=4$)

Also in this case only the QCP exists and it is, as usual, instructive to explore the asymptotic regimes $\omega_0/T \gg 1$ and $\omega_0/T \ll 1$.

(a) Regime $\omega_0/T \gg 1$

Here, the only solution of Eq. (4.59) is obtained from Eq. (4.60) setting $d=2$ to have for $h > h_c$ and $T \ll h-h_c$

$$\omega_0(T,h) \simeq (h-h_c) + \frac{2h_c}{S}\left(\frac{T}{\tau}\right)\ln^{-1}\left(\frac{\tau}{T}\right)e^{-(h-h_c)/T} \tag{4.65}$$

so that

$$\chi_\perp(T,h) \simeq \frac{2S}{h_c}\left(\frac{h-h_c}{h_c}\right)^{-1} \times\left\{1 - \frac{2}{S}\frac{h_c}{h-h_c}\left(\frac{T}{\tau}\right)\ln^{-1}\left(\frac{\tau}{T}\right)e^{-(h-h_c)/T}\right\}. \tag{4.66}$$

(b) Regime $\omega_0/T \ll 1$

In this regime, $H_1(\frac{\tau}{T},\frac{\omega_0}{T}) \simeq 2\ln\ln(\frac{\tau}{\omega_0})$ to leading order (see Appendix C) and Eq. (4.54) (for $d=2$) provides for $h < h_c$ and $T \ll h_c-h$

$$\omega_0(T,h) \simeq \tau\exp\left\{-\exp\left[\frac{S}{2}\left(\frac{\tau}{h_c}\right)\frac{h_c-h}{T}\right]\right\}, \tag{4.67}$$

and

$$\chi_\perp(T,h) \simeq \frac{2S}{\tau}\frac{h}{h_c}\exp\left\{\exp\left[\frac{S}{2}\left(\frac{\tau}{h_c}\right)\frac{h_c-h}{T}\right]\right\}. \tag{4.68}$$

A separate analysis is required in the regime at and around the QCP with $T \gg |h-h_c|$ where, as one can easily check, a solution of Eq. (4.54) exists only under the additional condition $\omega_0/T^2 \gg 1$. The suitable asymptotic solution to leading order in $|h-h_c|/T$ is

$$\omega_0(T,h) \simeq \frac{2h_c}{S\tau}\left(\frac{T}{\tau}\right)\frac{\ln\ln(\tau/T)}{\ln(\tau/T)} \times\left\{1 + \frac{S}{2}\left(\frac{\tau}{h_c}\right)2h_c\frac{\ln(\tau/T)}{\ln\ln(\tau/T)}\frac{h-h_c}{T}\right\}, \tag{4.69}$$

implying that

$$\chi_\perp(T,h) \simeq S^2\frac{h}{h_c^2}\left(\frac{\tau}{T}\right)\frac{\ln(\tau/T)}{\ln\ln(\tau/T)} \times\left\{1 - \frac{S}{2}\left(\frac{\tau}{h_c}\right)\frac{\ln(\tau/T)}{\ln\ln(\tau/T)}\frac{h-h_c}{T}\right\}. \tag{4.70}$$

In particular, for $h=h_c$ as $T \to 0$ along the QCT, we find $\omega_0(T,h) \simeq (2h_c/S\tau)T\ln\ln(\tau/T)/\ln(\tau/T)$ and $\chi_\perp \simeq S^2(\tau/h_c) T^{-1}\ln(\tau/T)/\ln\ln(\tau/T)$.

It is worth noting that also in the case $d=2$ with $\alpha=4$ the phase diagram appears quite similar to that shown in Fig. 4 but now only the crossover lines $T_1(h)$ and $T_2(h)$ exist.

(iii) $d > 2$ ($\alpha > 4$)

In this subregion a critical line, ending in the QCP, exists with the low-temperature representations (3.27) and (3.28).

As already shown for any d at the beginning of this subsection, in the regime $h > h_c$ and $\omega_0/T \gg 1$ the results (4.55) and (4.56) are true for $T \ll h-h_c$.

We now analyze the opposite regime $\omega_0/T \ll 1$ within the region of the phase diagram delimited by the phase boundary and the line $T_2(h) \simeq h-h_c$.

For this purpose, we first rewrite Eq. (4.54) for the energy gap in the convenient form

$$\frac{\omega_0}{T} \simeq \frac{g(T)}{T} + \frac{d}{2S}\left(\frac{h_c}{\tau}\right)\left(\frac{T}{\tau}\right)^{(d-2)/2} I_{d/2}\left(\frac{\tau}{T}, \frac{\omega_0}{T}\right), \quad (4.71)$$

where $g(T)=h-h_c(T)\geqslant 0$, $h_c(T)$ is given by Eq. (3.27) as $T\to 0$ and $I_{d/2}(\frac{\tau}{T},\frac{\omega_0}{T})=H_{d/2}(\frac{\tau}{T},\frac{\omega_0}{T})-H_{d/2}(\frac{\tau}{T},0)$, with $H_{d/2}(\frac{\tau}{T},0) \simeq f_{d/2}(\frac{\tau}{T})\simeq 2^{d/2}\Gamma(d/2)\ln^{-d/2}(\frac{\tau}{T})$ for $\frac{\tau}{T}\gg 1$.

For $h\leqslant h_c$, and only for these values of the applied field, we can write, in terms of $T_c(h)$ [see Eq. (3.28)], $g(T)\simeq d2^{(d-2)/2}\Gamma(d/2)\zeta(d/2)\frac{h_c}{S}[(\frac{T}{\tau}\ln^{-1}(\frac{\tau}{T}))^{d/2} -(\frac{T_c(h)}{\tau}\ln^{-1}(\frac{\tau}{T_c(h)}))^{d/2}]$ which measures the vertical distance from the critical line for $T\geqslant T_c(h)$ at fixed h.

To extract from Eq. (4.71) the quantum critical properties and crossovers close to the QCP, we consider separately the different intervals of dimensionalities, above $d=2$ along the line $\alpha=d+2$, where $I_{d/2}(\frac{\tau}{T},\frac{\omega_0}{T})$ has peculiar asymptotic expressions for $\frac{\tau}{T}\gg 1$ in the regime of interest $\omega_0/T\ll 1$ ($T\chi_\perp\gg 1$) (see appendix C).

(a) $2<d<4$ ($4<\alpha<6$)

For such dimensionalities, $I_{d/2}(\frac{\tau}{T},\frac{\omega_0}{T}) \simeq \frac{2^{d/2}\pi}{|\sin(\frac{\pi d}{2})|}(\frac{\omega_0}{T})^{(d-2)/2}\ln^{-d/2}(\frac{\tau}{\omega_0})$ and Eq. (4.71) has the asymptotic structure [to be compared with Eq. (4.29)]

$$\frac{\omega_0}{T} \simeq \frac{g(T)}{T} - \Omega\left(\frac{\tau}{\omega_0}\right), \quad (4.72)$$

where

$$\Omega\left(\frac{\tau}{\omega_0}\right) = \frac{d2^{(d-2)/2}\pi}{S\left|\sin\left(\frac{\pi d}{2}\right)\right|}\frac{h_c}{\tau}\left(\frac{\omega_0}{\tau}\right)^{(d-2)/2}\ln^{-d/2}\left(\frac{\tau}{\omega_0}\right). \quad (4.73)$$

With the basic criterion used for the subregion R_2 in Fig. 3, we now explore the two subregimes $\omega_0/T\ll\Omega$ and $\omega_0/T\gg\Omega$, with $\omega_0/T\simeq\Omega$ signaling the crossover between them, by approaching the critical line along horizontal [as $h\to h_c^+(T)$ at fixed T] and vertical [as $T\to T_c^+(h)$ at fixed $h\leqslant h_c$] trajectories.

For horizontal trajectories in the regime $\omega_0/T\ll\Omega$, Eq. (4.72) provides

$$\omega_0(T,h) \simeq \tau\left(\frac{1}{d-2}\right)^{d/(d-2)}\left[\frac{S\left|\sin\left(\frac{\pi d}{2}\right)\right|}{\frac{\pi d}{2}}\right]^{2/(d-2)} \times\left\{\left(\frac{\tau}{T}\right)\frac{h-h_c(T)}{h_c}\right\}^{2/(d-2)} \times\ln^{d/(d-2)}\left\{\left(\frac{\tau}{T}\right)\frac{h-h_c(T)}{h_c}\right\}, \quad (4.74)$$

and hence focusing as usual on the transverse susceptibility,

$$\chi_\perp(T,h) \simeq \frac{2S}{\tau}\frac{h}{h_c}\left(\frac{1}{d-2}\right)^{-d/(d-2)}\left[\frac{S\left|\sin\left(\frac{\pi d}{2}\right)\right|}{\frac{\pi d}{2}}\right]^{-2/(d-2)} \times\left\{\left(\frac{\tau}{T}\right)\frac{h-h_c(T)}{h_c}\right\}^{-2/(d-2)} \times\ln^{-d/(d-2)}\left\{\left(\frac{\tau}{T}\right)\frac{h-h_c(T)}{h_c}\right\}. \quad (4.75)$$

The power-law parts are obtained from Eqs. (4.31) and (4.32) performing the limit as $\alpha\to(d+2)^-$ but the presence of logarithmic corrections implies an exotic CC regime. In the opposite subregime $\omega_0/T\gg\Omega$, we find the MF result $\omega_0\simeq h-h_c(T)$ without logarithmic corrections. The crossover between these subregimes is characterized by the Ginzburg-like line for horizontal trajectories

$$h_{Gi}(T) \simeq h_c(T) + (4-d)^{d/(4-d)}\left[\frac{\pi d/2}{S\left|\sin\left(\frac{\pi d}{2}\right)\right|}\right]^{2/(4-d)} h_c\left(\frac{h_c}{\tau}\right)^{(d-2)/(4-d)}\left(\frac{T}{\tau}\right)^{2/(4-d)}\ln^{-d/(4-d)}\left(\frac{\tau}{T}\right). \quad (4.76)$$

Notice that, as $d\to 4^-$, $h_{Gi}(T)\to h_c^+(T)$ and the regime $\omega_0/T\ll\Omega$ disappears.

For vertical trajectories with $T\to T_c^+(h)\neq 0$ (for $h<h_c$) and $g(T)\propto(\frac{T-T_c(h)}{T_c(h)})$, Eq. (4.72) provides the non-conventional result for transverse susceptibility

$$\chi_\perp(T,h) \simeq \begin{cases} 2S\frac{h}{h_c}\left(\frac{1}{d-2}\right)^{-d/(d-2)}\left[\frac{1}{\pi}\left|\sin\left(\frac{\pi d}{2}\right)\right|\frac{d}{2}\Gamma\left(\frac{d}{2}\right)\zeta\left(\frac{d}{2}\right)\right]^{-2/(d-2)} \\ \times T_c^{-1}(h)\ln^{d/(d-2)}\left(\frac{\tau}{T_c(h)}\right)\left(\frac{T-T_c(h)}{T_c(h)}\right)^{-2/(d-2)}\ln^{-d/(d-2)}\left(\frac{T_c(h)}{T-T_c(h)}\right), & \frac{\omega_0}{T}\ll\Omega \\ \frac{(2S/d)^2}{2^{d/2}\Gamma(d/2)\zeta(d/2)}\frac{h}{h_c^2}\left(\frac{\tau}{T_c(h)}\right)^{d/2}\ln^{d/2}\left(\frac{\tau}{T_c(h)}\right)\left(\frac{T-T_c(h)}{T_c(h)}\right)^{-1}, & \frac{\omega_0}{T}\gg\Omega. \end{cases} \quad (4.77)$$

The crossover between the exotic CC regime and the MF one is signaled by $\frac{\omega_0}{T} \simeq \Omega$ which defines the thermal Ginzburg-like line within the region $h < h_c$ of the phase diagram [to be compared with Eq. (4.36) in the limit $\alpha \to (d+2)^-$]

$$T_{Gi}(h) \simeq T_c(h) + (4-d)^{d/(4-d)}\left[\frac{\pi d/2}{\left|\sin\left(\frac{\pi d}{2}\right)\right| S}\left(\frac{h_c}{\tau}\right)\right]^{2/4-d}\left(\frac{2}{d}\right)^2 \times \frac{S\tau^2/h_c}{2^{d/2}\Gamma(d/2)\zeta(d/2)}\left(\frac{T_c(h)}{\tau}\right)^{\{(d-2)^2/[2(4-d)]\}+1} \times \ln^{-(d-2)/[2(4-d)]}\left(\frac{\tau}{T_c(h)}\right). \quad (4.78)$$

When $h = h_c(T_c(h_c) = 0)$, with $g_c(T) = d2^{(d-2)/2}\Gamma(d/2)\zeta(d/2)\frac{h_c}{S}(\frac{T}{\tau})^{d/2}\ln^{-d/2}(\frac{\tau}{T})$ and sufficiently close to the QCP, Eq. (4.72) provides the asymptotic energy gap solution

$$\omega_0(T,h_c) \simeq d2^{(d-2)/2}\Gamma(d/2)\zeta(d/2)\frac{h_c}{S}\left(\frac{T}{\tau}\right)^{d/2}\ln^{-d/2}\left(\frac{\tau}{T}\right). \quad (4.79)$$

So, decreasing T along the QCT, one finds $\chi_\perp(T,h_c) \propto T^{d/2}\ln^{-d/2}(\frac{1}{T})$ in contrast with the conventional ψ-renormalized power-law behavior.[15,20,56]

In the remaining region of the phase diagram for $h > h_c$ and $T \gg T_2(h) \simeq h - h_c$ the behavior is again governed by Eq. (4.72) with $g(T) \simeq (h-h_c) + d2^{(d-2)/2}\frac{h_c}{S}\Gamma(d/2)\zeta(d/2)(\frac{T}{\tau})^{d/2}\ln^{-d/2}(\frac{\tau}{T})$. It is then easy to argue that the line $T_1(h) \simeq \tau(\frac{2}{d})[\frac{d}{S}2^{(d-2)/2}\Gamma(d/2)\zeta(d/2)]^{-2/d}(\frac{h-h_c}{h_c})^{2/d}\ln(\frac{h_c}{h-h_c})$, symmetric to the critical line (3.28) with respect to $h = h_c$, signals the crossover between a regime [for $T \gg T_1(h)$] where the thermodynamic quantities of interest are essentially independent of $h - h_c$ and behave as along the QCT, to another one [for $T_2(h) \ll T \ll T_1(h)$] with $\omega_0(T,h) \simeq h - h_c$ except for small corrections of the type $(\frac{T}{\tau})^{d/2}\ln^{-d/2}(\frac{\tau}{T})$ and $(\frac{T}{\tau})(\frac{h-h_c}{h_c})^{(d-2)/2}\ln^{-d/2}(\frac{h_c}{h-h_c})$.

(b) $d \geq 4$ ($\alpha \geq 6$)

In this range of dimensionalities $I_{d/2}(\frac{\tau}{T}, \frac{\omega_0}{T}) \simeq -2^{d/2}\Gamma(d/2)\zeta(\frac{d-2}{2})\frac{\omega_0}{T}\ln^{-d/2}(\frac{\tau}{T})$ in Eq. (4.71) for $\omega_0/T \ll 1$ and hence in Eq. (4.72) we have $\Omega \simeq \frac{d2^{(d-2)/2}}{S}\Gamma(d/2)\zeta(\frac{d-2}{2})(\frac{h_c}{\tau})(\frac{T}{\tau})^{(d-2)/2}\frac{\omega_0}{T}\ln^{-d/2}(\tau/T)$. Thus, we get $\frac{\omega_0/T}{\Omega} \simeq \frac{S(\tau/h_c)}{d2^{(d-2)/2}\Gamma(d/2)\zeta(\frac{d-2}{2})}(\frac{T}{\tau})^{-(d-2)/2}\ln^{d/2}(\frac{\tau}{T}) \gg 1$ for $\tau/T \gg 1$ and the regime $\omega_0/T \ll \Omega$ is absent [no Ginzburg-like line exists in the phase diagram (see Fig. 5)].

For horizontal trajectories, we have trivially the MF behaviors $\omega_0(T,h) \simeq h - h_c(T)$ and $\chi_\perp(T,h) \simeq 2S(\frac{h}{h_c})(h-h_c(T))^{-1}$ as $h \to h_c^+(T)$.

For vertical trajectories, a scenario similar to that found for $2 < d < 4$ takes place with $g(T) \simeq (h-h_c) + d2^{(d-2)/2}\frac{h_c}{S}\Gamma(d/2)\zeta(d/2)(\frac{T}{\tau})^{d/2}\ln^{-d/2}(\tau/T)$ or, in terms of $T_c(h)$ for $h \leq h_c$, $g(T) \simeq d2^{(d-2)/2}\Gamma(d/2)\zeta(d/2)\frac{h_c}{S}[(\frac{T}{\tau})^{d/2}\ln^{-d/2}(\frac{\tau}{T}) - (\frac{T_c(h)}{\tau})^{d/2}\ln^{-d/2}(\frac{\tau}{T_c(h)})]$.

At fixed $h < h_c$, $g(T) \propto (T - T_c(h))$ as $T \to T_c^+(h) \neq 0$ and we easily get

$$\omega_0(T,h) \simeq \frac{d^2 2^{(d-4)/2}h_c}{S}\Gamma(d/2)\zeta(d/2)\left(\frac{T_c(h)}{\tau}\right)^{d/2}\ln^{-d/2}\left(\frac{\tau}{T_c(h)}\right) \times \left(\frac{T-T_c(h)}{T_c(h)}\right), \quad (4.80)$$

and the conventional MF behavior (also for $d=4$) of the transverse susceptibility

$$\chi_\perp(T,h) \simeq \frac{2S^2/h_c}{d^2 2^{(d-4)/2}\Gamma(d/2)\zeta(d/2)}\left(\frac{h}{h_c}\right) \times \left(\frac{T_c(h)}{\tau}\right)^{-d/2}\ln^{d/2}\left(\frac{\tau}{T_c(h)}\right)\left(\frac{T-T_c(h)}{T_c(h)}\right)^{-1}, \quad (4.81)$$

implying absence of logarithmic corrections in $T - T_c(h)$ for $d=4$ (in contrast with that expected for SRI's). Notice that these results are formally identical to those found for $2 < d < 4$ in the regime $\omega_0/T \gg \Omega$ [see Eq. (4.77)]. Along the QCT we have

$$\omega_0(T,h_c) \simeq \frac{d2^{(d-2)/2}}{S}\Gamma(d/2)\zeta(d/2)\left(\frac{T}{\tau}\right)^{d/2}\ln^{-d/2}\left(\frac{\tau}{T}\right), \quad (4.82)$$

so that, approaching the QCP,

$$\chi_\perp(T,h_c) \simeq \frac{S^2/h_c}{d2^{(d-4)/2}\Gamma(d/2)\zeta(d/2)}\left(\frac{T}{\tau}\right)^{-d/2}\ln^{d/2}\left(\frac{\tau}{T}\right), \quad (4.83)$$

which is exactly the result (4.79) found for $2 < d < 4$.

Finally, in the region $h > h_c$ and $T \gg T_2(h) \simeq h - h_c$, we find that for $T \gg T_1(h)$, the line symmetric to $T_c(h)$ with respect to the vertical line $h = h_c$, the spin model exhibits the same quantum critical behavior found before along the QCT; in the intermediate temperature regime for $T_2(h) \ll T \ll T_1(h)$, we obtain

$$\omega_0(T,h) \simeq (h-h_c) + \frac{d2^{(d-2)/2}}{S}h_c\Gamma(d/2)\zeta(d/2) \times \left(\frac{T}{\tau}\right)^{d/2}\ln^{-d/2}\left(\frac{\tau}{T}\right), \quad (4.84)$$

and

$$\chi_\perp(T,h) \simeq \frac{2S}{h_c}\left(\frac{h}{h_c}\right)\left(\frac{h-h_c}{h_c}\right)^{-1} \times\left\{1-\frac{d2^{(d-2)/2}\Gamma(d/2)\zeta(d/2)}{S}\left(\frac{h_c}{h-h_c}\right) \times\left(\frac{T}{\tau}\right)^{d/2}\ln^{-d/2}\left(\frac{\tau}{T}\right)\right\}. \quad (4.85)$$

Notice that in this last region of the phase diagram, for $d=4$, the transverse susceptibility exhibits a correction in temperature as $T^2\ln(1/T)$ to the MF behavior in terms of the $h-h_c$ which occurs for $T \ll T_2(h)$.

In conclusion, in the boundary situation $\alpha=d+2$, although the V-shaped structure of the phase diagram close to the QCP remains qualitatively similar to that shown in Fig. 5, the low-temperature behavior for any d appears quite peculiar due to the competition of the fluctuational LRI and SRI regimes in the (α,d) plane. In particular, logarithmic corrections to critical power-law behaviors emerge where they are unexpected.

V. CROSSOVER SCALING FUNCTIONS AND EFFECTIVE CRITICAL EXPONENTS WITHIN THE SUBREGION R_2 OF THE (α,d) PLANE

All the crossover behaviors which occur close to the QCP when (α,d) range within the region R_2 of the (α,d) plane, where only power-law behaviors are present, can be properly described by scaling arguments, consistently with the basic ideas of the general crossover theory.[59] Due to the relation $\chi_\perp=2S\sigma/\omega_0$ (with $\sigma\sim 1$ close to the QCP) it is convenient to work in terms of energy gap ω_0. We will show below, by explicit calculation, that the full solution of the self consistent equation (4.28) can be obtained in the scaling form.[38,60–62]

$$\omega_0 \simeq g(T)Z(z), \quad (5.1)$$

where

$$z=\frac{\Lambda_S[g(T)]^{\varphi}}{T}, \quad (5.2)$$

and the scaling function $Z(z)$, the crossover exponent φ and the constant Λ_S have to be determined. First, from the known asymptotic behaviors of ω_0 for horizontal trajectories (see Sec. IV A 3), it is easy to check that $Z(z)$ must satisfy the boundary conditions

$$Z(z)\simeq\begin{cases} z^{\gamma_h}, & z\ll 1\\ 1, & z\gg 1,\end{cases} \quad (5.3)$$

with $\gamma_h=\frac{\alpha-d}{2d-\alpha}, \varphi=\frac{2\alpha-3d}{\alpha-d}$,and

$$\Lambda_S=\frac{\left|\sin\left(\frac{\pi d}{\alpha-d}\right)\right|S\tau^{d/(\alpha-d)}}{h_c[\pi d/(\alpha-d)]}.$$

Then, a suitable equation for the unknown scaling function $Z(z)$ can be obtained setting $\omega_0/g(T)\simeq Z(z)$ in Eq. (4.28) and then assuming $Z(z)$ and z fixed in the low-temperature regime.[60–62] A simple algebra provides

$$Z(z)+z^{-1}[Z(z)]^{(2d-\alpha)/(\alpha-d)}=1, \quad (5.4)$$

or, equivalently,

$$Z(z)z^{1/\varphi}+[Z(z)z]^{1/\gamma_h}-z^{1/\varphi}=0. \quad (5.5)$$

Of course, this equation has to be solved numerically for arbitrary values of (α,d) within the subregion R_2.

The problem simplifies sensibly for particular cases which may be also of experimental interest.

For $d=1$ $(3/2<\alpha<2)$ and $\alpha=9/5=1.8$, we have $\varphi=3/4$, $\gamma_h=4$ and, setting $Y(z)=[z^{4/3}Z(z)]^{1/4}$ we get the quartic equation

$$Y^4(z)+Y(z)-z^{4/3}=0, \quad (5.6)$$

which should be, at least in principle, resoluble. When the appropriate solution has been found, we should have $Z(z)=z^{-4/3}[Y(z)]^4$.

In the case $d=2$ $(3<\alpha<4)$ and $\alpha=10/3=3.33,\ldots$, it is $\varphi=1/2$, $\gamma_h=2$, and with $Y(z)=[z^2Z(z)]^{1/2}$, we find the quadratic equation

$$Y^2(z)+Y(z)-z^2=0, \quad (5.7)$$

which can be easily solved. This allows us to obtain $Z(z)$ as

$$Z(z)=\{[(2z)^2+1]^{1/2}-1\}^2/(2z)^2. \quad (5.8)$$

Finally, for $d=3$ $(9/2<\alpha<5)$ and $\alpha=24/5=4.8$ with $\varphi=1/3$ and $\gamma_h=3/2=1.5$, setting $Y(z)=[z^3Z(z)]^{1/3}$ we have the cubic equation

$$Y^3(z)+Y^2(z)-z^3=0, \quad (5.9)$$

which can be solved to obtain the suitable solution such that $Z(z)=z^{-3}Y3(z)$ satisfies boundary conditions (5.3).

For generic values of (α,d) in the subregion of interest, we can also obtain explicit analytic expressions of $Z(z)$ for small and large values of the scaling variable z solving Eq. (5.4) or (5.5) for ascending powers of z and z^{-1} respectively.[60,62] To leading orders one obtains

$$Z(z)\simeq\begin{cases} z^{\gamma_h}\left\{1-\gamma_h z^{\gamma_h}+\frac{1}{2}\gamma_h(3\gamma_h-1)z^{2\gamma_h}+\cdots\right\}, & z\ll 1\\ 1-z^{-1}+\frac{1}{\gamma_h}z^{-2}+\cdots, & z\gg 1.\end{cases} \quad (5.10)$$

In any case, when $Z(z)$ is obtained as a function of $z=z(T,h)$ one can easily derive the suitable scaling functions for other relevant thermodynamic quantities expressed in terms of the energy gap $\omega_0(T,h)$. For instance, using Eq. (5.1), close to the QCP we have

$$\chi_\perp \simeq 2Sg^{-1}(T)X \times\left(\frac{T}{\Lambda_s[g(T)]^{\varphi}}\right), \quad (5.11)$$

where $X(x)\equiv Z^{-1}(z)$ with $x=z^{-1}$.

All the near QCP crossovers can be more expressively described in terms of the effective critical exponent for transverse susceptibility defined by $\gamma_h^{eff}(z)=-\frac{\partial \ln \chi_\perp}{\partial \ln g(T)}$. In terms of the scaling function $Z(z)$ we easily have the representation

$$\gamma_h^{eff}(z) = 1 + \frac{2\alpha - 3d}{\alpha - d} z \frac{d}{dz} \ln Z(z). \tag{5.12}$$

For dimensionalities $d=1,2,3$ and suitable values of α, one can obtain explicit analytical expression for $\gamma_h^{eff}(z)$. For instance, for $d=2$ and $\alpha=10/3$, Eq. (5.8) provides

$$\gamma_h^{eff}(z) = \frac{4z^2}{(1+4z^2)-(1+4z^2)^{1/2}}. \tag{5.13}$$

One can immediately check that this describes the full crossover between the MF regime ($\gamma_h=1$) and the CC one ($\gamma_h=2$) along horizontal trajectories as $g(T)\to 0^+$, as expected in view of the general asymptotic findings of Sec. IV. For generic values of (α,d) in the subregion R_2, using the asymptotic expressions (5.10) for $Z(z)$, simple algebra yields

$$\gamma_h^{eff}(z) \simeq \begin{cases} \dfrac{\alpha-d}{2d-\alpha}\left\{1 - \dfrac{2\alpha-3d}{2d-\alpha} z + \dfrac{(2\alpha-3d)(3\alpha-4d)}{2d-\alpha} z^2 + \cdots \right\}, & z \ll 1 \\ 1 + \dfrac{2\alpha-3d}{\alpha-d} z^{-1} - \dfrac{(2\alpha-3d)(5d-3\alpha)}{(\alpha-d)^2} z^{-2} + \cdots, & z \gg 1. \end{cases} \tag{5.14}$$

It is worth nothing that, in terms of the same scaling function $Z(z)$, one can describe also the crossovers which occurs approaching the critical line along vertical trajectories, i.e., as $T\to T_c^+(h)$ at fixed $h\leq h_c$. This can be simply performed using for $g(T)$ and z the representations

$$g(T) = \frac{d}{S(\alpha-d)} F_{d/(\alpha-d)}(0) h_c (1-\tau_1^{d/(\alpha-d)}) \tau_2^{d/(\alpha-d)}, \tag{5.15}$$

and

$$z(\tau_1,\tau_2) = C_S(\alpha,d)(1-\tau_1^{d/(\alpha-d)})^{(2\alpha-3d)/(\alpha-d)} \tau_2^{-[(2d-\alpha)/(\alpha-d)]^2}, \tag{5.16}$$

in terms of the dimensionless parameters $\tau_1=\frac{T_c(h)}{T}$ and $\tau_2=\frac{T}{\tau}$.

Here $C_S(\alpha,d)=(\frac{|\sin(\frac{\pi d}{\alpha-d})|}{\pi})(\frac{h_c d}{(\alpha-d)S\tau})^{-(2d-\alpha)/(\alpha-d)} \times[F_{d/(\alpha-d)}(0)]^{(2\alpha-3d)/(\alpha-d)}$, $0\leq\tau_1\leq 1$ and $\tau_2\geq 0$, with $\tau_1=0$ for $h=h_c$ along the QCT.

With these ingredients, the energy gap can be written as

$$\omega_0(\tau_1,\tau_2) = \frac{d}{S(\alpha-d)} F_{d/(\alpha-d)}(0) h_c (1-\tau_1^{d/(\alpha-d)}) \times \tau_2^{-d/(\alpha-d)} Z(z(\tau_1,\tau_2)), \tag{5.17}$$

and hence, for the transverse susceptibility, we have close to the QCP

$$\chi_\perp \simeq 2S\left[\frac{d}{S(\alpha-d)} F_{d/(\alpha-d)}(0) h_c\right]^{-1} \times (1-\tau_1^{d/(\alpha-d)})^{-1} \tau_2^{d/(\alpha-d)} Z^{-1}(z(\tau_1,\tau_2)). \tag{5.18}$$

Defining the related effective critical exponent for vertical trajectories as $\gamma_T^{eff}=-\frac{\partial \ln \chi_\perp}{\partial \ln(T-T_c(h))}$ to include the possibility $T_c(h)\to 0$ for $h\to h_c^-$, Eqs. (5.15), (5.16), and (5.18), in terms of τ_1 and τ_2, provide

$$\gamma_T^{eff}(\tau_1,\tau_2) = \frac{\alpha-d}{2\alpha-3d}(1-\tau_1) \times\left\{1 - \left[1 - \frac{d(2\alpha-3d)}{(\alpha-d)^2(1-\tau_1^{d/(\alpha-d)})}\right] \times \gamma_h^{eff}(z(\tau_1,\tau_2))\right\}. \tag{5.19}$$

For the QCT ($\tau_1=0$), this reduces to

$$\gamma_T^{eff}(0,\tau_2) = \frac{\alpha-d}{2\alpha-3d}\left\{1 - \left(\frac{2d-\alpha}{\alpha-d}\right)^2 \gamma_h^{eff}(z(0,\tau_2))\right\}. \tag{5.20}$$

For instance, in the case $d=2$ and $\alpha=10/3$, we have simply

$$\gamma_T^{eff}(\tau_1,\tau_2) = 2(1-\tau_1)\left\{1 - \left[1 - \frac{3}{4(1-\tau_1^{3/2})}\right] \gamma_h^{eff}(C_S(10/3,2) \times (1-\tau_1^{3/2})^{1/2} \tau_2^{-1/4})\right\}, \tag{5.21}$$

and

$$\gamma_T^{eff}(0,\tau_2) = 2\left\{1 - \frac{1}{4}\gamma_h^{eff}(C_S(10/3,2)\tau_2^{-1/4})\right\}. \tag{5.22}$$

Previous results constitute the general solution to our problem close to the QCP for $h\leq h_c$. In particular, one can easily check[38] that the expressions obtained for γ_T^{eff} allow to describe (consistently) all the crossovers which occur in the disordered phase decreasing T at fixed $h\leq h_c$, reproducing all the asymptotic results directly obtained in Sec. IV solving the self-consistent equation for ω_0 in different regimes in the vicinity of the QCP. Corrections to the asymptotic results can be simply obtained by using Eq. (5.14).

VI. REMARKS ON QUANTUM CRITICAL DYNAMICS AND CONCLUSIONS

In the previous sections we have used the two-time GF method at a RPA level to study, in a systematic way, the static quantum criticality of a d-dimensional spin-S easy-plane Heisenberg ferromagnet with LRIs decaying as $r^{-\alpha}$ with the distance r between the spin sites in the presence of a longitudinal magnetic field. The structure of the low-temperature phase, diagram and a rich crossover scenario in the (h,T) plane close to the field-induced QCP have been found by variation of the interaction parameter α and the dimensionality d. To this lowest order of approximation, which is however expected to provide reliable physical results in the low-temperature regime,[37] focusing on the region $d<\alpha<d+2$ of the (α,d) plane where the LRI's are effective, we have estimated the phase boundary and the main quantum critical properties of the model have been obtained. For case $\alpha>d+2$, we have shown that the TD procedure reproduces the known QCP scenario already found for an exactly solvable easy-plane Heisenberg ferromagnetic model with in-plane SRIs and uniform ones in the longitudinal direction[38,53] and, for the most realistic model (2.1) with spin $1/2$ and full short-range exchange interactions, by using the RG approach.[18,20] This features suggest that, very close to the QCP, the TD may provide near-exact results, although it gives incorrect critical exponents by approaching the critical line (when it exists) at finite temperature. Remarkably, for appropriate values of α, also in low dimensions we can have a critical line ending in the QCP, circumstance which may be of interest for the low-temperature physics of complex magnets.[19] Moreover, since several systems (as the superfluid ones) may be mapped into the anisotropic spin model (2.1) with $S=1/2$,[19,42–47] our predictions about the transverse quantum criticality in the (α,d) plane may be usefully employed for giving possible insight also into the low-temperature properties of these experimentally interesting systems. Before concluding the paper, we wish to note that, within the same framework, one has all the ingredients to explore promptly also the quantum critical dynamics by variation in α and d. In particular, for the region $d<\alpha<d+2$ of the (α,d) plane where the LRIs are effective and the energy spectrum deviates from the gap value ω_0 as a genuine power law in k as $k\to 0$ [see Eq. (3.16)], relevant information can be extracted from the scaling structure of the transverse dynamic susceptibility $\chi_\perp(\mathbf{k},\omega)=-G(\mathbf{k},\omega)$ for small values of its arguments. Indeed, combining Eqs. (2.20) and (3.16), we find for transverse dynamic susceptibility the following scaling representation:

$$\chi_\perp(\mathbf{k},\omega)\simeq \xi_\perp^{(\alpha-d)}[SJA_d((k\xi_\perp)^{(\alpha-d)}+1)-\xi_\perp^{(\alpha-d)}\omega]^{-1}, \quad (6.1)$$

where $\xi_\perp\simeq(SJAd)^{1/(\alpha-d)}\chi_\perp^{1/(\alpha-d)}\sim\chi_\perp^{1/(\alpha-d)}$ defines the transverse correlation length and the related critical exponent $\nu=\gamma/(\alpha-d)$ for case of LRIs.

Then, from a comparison with the general dynamic scaling relation $\chi(\mathbf{k},\omega)=\xi^{2-\eta}W(k\xi,\xi^z\omega)$, we immediately have

$$\eta=(d+2)-\alpha,\quad z=2-\eta=\alpha-d, \quad (6.2)$$

for the Fisher η and dynamic z critical exponents, and $W(x,y)=[SJA_d(x^{\alpha-d}+1)-y]^{-1}$.

When $\alpha>d+2$, Eq. (3.16) provides $\eta=0$, $z=2$ and $W(x,y)=[SJC_d(x^2+1)-y]^{-1}$, as expected.[18,20,38,53]

In conclusion, the rich scenario obtained in the present paper in a unified and consistent way for the quantum nontrivial anisotropic spin model (2.1) with $K(0)<J(0)$ supports the effectiveness of the Green's function method already at a RPA level to capture the essential physics close to the field-induced QCP for arbitrary values of dimensionality d and the interaction decaying exponent $\alpha>d$. It appears quite consistent with the RG treatments,[18,20] for case of short-range interactions, and exact findings[38,53,63,64] for simplified spin models with longitudinal uniform interactions. It is also worth emphasizing that the rather simple decoupling procedure used here appears adequate to capture the essential low-temperature physical features of the anisotropic spin model under study in the influence domain of the QCP. An exception is the region very near the phase boundary at finite temperature (when it exists) where incorrect spherical model-like critical exponents are found. Previous statement is well supported by the remarkable consistency of our predictions around the QCP beyond the *CC* region (see Fig. 5) with those obtained on the ground of the RG applied to the functional representation of the spin model (2.1) with $S=1/2$ and SRIs (Refs. 18 and 40) and for certain exactly solvable anisotropic magnetic models.[38,53,63,64] Finally, we would like to underline that some methodological aspects of the unified approach, used through this paper for exploring quantum criticality in the two-time GF framework, may be a useful guide for other higher order studies of more realistic magnetic models which exhibit a field-induced quantum critical point.

APPENDIX A: FOURIER TRANSFORMS FOR POWER-LAW LONG-RANGE EXCHANGE INTERACTIONS

The Fourier transform $X(\mathbf{k})/X$ of $X_{ij}/X=r_{ij}^{-\alpha}$ has the representation[30,32,33]

$$\frac{X(\mathbf{k})}{X}=\frac{X(0)}{X}+(2\pi)^{d/2}\left\{\frac{\Gamma\left(-\frac{\alpha-d}{2}\right)}{2^{\alpha-d/2}\Gamma\left(\frac{\alpha}{2}\right)}k^{\alpha-d}-2^{1-d/2}\times\sum_{l=1}^{\infty}\frac{(-1)^l}{4^l l![2l-(\alpha-d)]\Gamma(l+d/2)}\frac{k^{2l}}{\Lambda_{1BZ}^{2l-(\alpha-d)}}\right\} \quad (A1)$$

for $\alpha>d$ and $\alpha\neq d+2n$ $(n=1,2,\ldots)$, and

$$\frac{X(\mathbf{k})}{X}=\frac{X(0)}{X}-\frac{\pi^{d/2}}{d\Gamma\left(\frac{d}{2}\right)}k^2\ln\left(\frac{\Lambda_{1BZ}}{k}\right)-\pi^{d/2}\times\sum_{l=2}^{\infty}\frac{(-1)^l}{4^l l!(l-1)\Gamma(l+d/2)}\frac{k^{2l}}{\Lambda_{1BZ}^{2l-2}} \quad (A2)$$

for $\alpha=d+2$.

APPENDIX B: ASYMPTOTIC EXPANSIONS FOR THE BOSE-EINSTEIN FUNCTION

For the Bose-Einstein function $F_\nu(y)$ [Eq. (4.3)] the following general expansions in $y \geqslant 0$ are true: [57,58]

$$F_\nu(y) = \begin{cases} \Gamma(\nu)\Gamma(1-\nu)y^{\nu-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\Gamma(\nu)\zeta(\nu-k)y^k, & \nu \neq 1,2,3,\ldots \\ (-1)^{\nu-1}y^{\nu-1}\left[-\ln y + \sum_{m=1}^{\nu-1}\frac{1}{m}\right] + \sum_{k=0,k\neq\nu-1}^{\infty} \frac{(-1)^k}{k!}\Gamma(\nu)\zeta(\nu-k)y^k, & \nu = 1,2,3,\ldots. \end{cases} \tag{B1}$$

Here $\Gamma(z)$ is the gamma function, $\zeta(\nu-k)$ for any k is obtained by analytical continuation of the Riemann zeta-function $\zeta(s)=\Sigma_{k=1}^{\infty}k^{-s}$, and the convention $(\Sigma_{m=1}^{\nu-1}1/m)_{\nu=1}=0$ is assumed.

For $y \gg 1$ we have, for any $\nu > 0$, the asymptotic behavior

$$F_\nu(y) \approx \Gamma(\nu)e^{-y}, \tag{B2}$$

while, for $y \ll 1$, different y behaviors take place, depending on the values of ν as specified below

$$F_\nu(y) \simeq \begin{cases} \frac{\pi}{\sin(\pi\nu)}y^{\nu-1} + O(y^0), & 0 < \nu < 1 \\ \ln\frac{1}{y} + O(y), & \nu = 1 \\ \Gamma(\nu)\zeta(\nu) + \frac{\pi}{\sin(\pi\nu)}y^{\nu-1} + O(y), & 1 < \nu < 2 \\ \frac{\pi^2}{6} - y\ln\frac{1}{y} + O(y), & \nu = 2 \\ \Gamma(\nu)\zeta(\nu) - (\nu-1)\Gamma(\nu-1)\zeta(\nu-1)y + O(y^2), & \nu > 2. \end{cases} \tag{B3}$$

For $\nu=1$, the integral $F_1(\nu)$ can be performed exactly for any $y \geqslant 0$, to have

$$F_1(y) = \int_0^\infty \frac{dx}{e^{x+y}-1} = \ln\frac{1}{1-e^{-y}}. \tag{B4}$$

This provides

$$F_1(y) \simeq \begin{cases} e^{-y}, & y \gg 1 \\ \ln\frac{1}{y}, & y \ll 1, \end{cases} \tag{B5}$$

in agreement with the asymptotic expansions (B2) and (B3).

APPENDIX C: ASYMPTOTIC BEHAVIORS OF THE INTEGRAL $H_\nu(\delta,y)$ FOR CASE $\alpha=d+2$

Let us consider the two-parameter integral [see Eq. (4.53)]

$$H_\nu(\delta,y) = \int_0^\delta dx \frac{x^{\nu-1}}{\exp\left[y + \frac{1}{2}x\ln(\delta/ax)\right]-1}, \tag{C1}$$

which enters the critical properties of the spin model (2.1) in the borderline interaction regime $\alpha=d+2$. Here $\delta=\tau/T \gg 1$ in all situations of interest, $y=\omega_0/T \geqslant 0$, $a=\Lambda_{1BZ}^2 > 0$ is a positive constant and $\nu=d/2$. We are interested to its different asymptotic behaviors for $\delta \gg 1$ and large or small values of y.

By using a suitable variable change, integral (C1) can be rewritten in the most convenient form

$$H_\nu(\delta,y) = 2^\nu \int_0^{(\delta/2)\ln(1/a)} dx \frac{1}{\ln^\nu\left(\frac{\delta}{x}\right)} \frac{x^{\nu-1}}{e^{x+y}-1}. \tag{C2}$$

First, it is straightforward to check that, for $y\delta \gg 1$, Eq. (C2) provides

$$H_\nu(\delta,y) \simeq 2^\nu \frac{F_\nu(y)}{\ln^\nu(\delta)} \tag{C3}$$

for any $\nu > 0$, where $F_\nu(y)$ has been discussed in Appendix B.

For $y \ll 1$ we find the asymptotic behaviors

$$H_\nu(\delta,y) \simeq \begin{cases} 2^\nu \dfrac{\pi}{\sin(\pi\nu)} \dfrac{y^{\nu-1}}{\ln^\nu(\delta/y)}, & 0<\nu<1 \quad \text{(C4a)} \\ 2\ln\ln(\delta/y), & \nu=1 \quad \text{(C4b)} \\ 2^\nu \dfrac{\Gamma(\nu)\zeta(\nu)}{\ln^\nu(\delta)} + \left[\begin{array}{ll} 2^\nu \dfrac{\pi}{\sin(\pi\nu)} \dfrac{y^{\nu-1}}{\ln^\nu(\delta/y)}, & 1<\nu<2 \quad \text{(C4c)} \\ -2^2 y/\ln(\delta), & \nu=2 \quad \text{(C4d)} \\ -2^\nu(\nu-1)\Gamma(\nu-1)\zeta(\nu-1)\dfrac{y}{\ln^\nu(\delta)}, & \nu>2 \quad \text{(C4e)} \end{array}\right. \end{cases}$$

Results (C3), (C4d), and (C4e) are obtained assuming $\ln(\delta/x) \simeq \ln(\delta)$ in the integrand of (C2) since the region $x > 1/\delta$ provides the main contribution to the integral. Similarly, the leading behaviors (C4a) and (C4c) follow setting $\ln(\delta/x) \simeq \ln(\delta/y)$ in the region $x > y^2/\delta$, while (C4b) arises from the region $x > y$ where one can simply assume $e^{x+y} \simeq e^x$.

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